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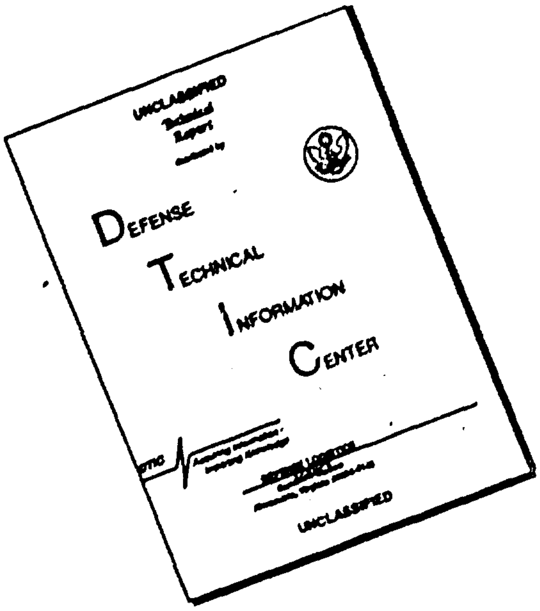

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STATISTICAL TECHNIQUES IN LIFE TESTING

CHAPTER II

TESTING OF HYPOTHESES

by

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AUTHOR'S NOTE

CHAPTER II, "Testing of Hypotheses," is part of the material being prepared in connection with a contemplated handbook or monograph on Statistical Techniques in Life Testing. It is in the nature of a preliminary report on one aspect of the over-all undertaking.

Earlier reports were:

Technical Report No. 1, "Statistical Developments in Life Testing," June 1, 1957.

Technical Report No. 2, "The Exponential Distribution and Its Role in Life Testing," May 1, 1958.

"An Outline of Three Chapters of a Handbook on Statistical Methods in Life Testing," June 6, 1958.

Further material dealing with other aspects of life testing is in preparation.

Comments and suggestions are invited.

Benjamin Epstein

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TESTING OF HYPOTHESES

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Chapter II

Testing of Hypotheses

Introductory remarks: In the following we shall assume that the underlying p.d.f. of the life-time X is described by

$$(1) \qquad f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \; , \; x > 0 \; , \; \theta > 0 \; .$$

Our object is to test various hypotheses regarding the mean life θ on the basis of censored, truncated, or sequential procedures. Various tables and graphs are given and in the appendix, appropriate references and proofs appear.

Section 1

Problem: Give a censored life test procedure which will have the property that the probability of rejecting a lot with mean life $\theta = \theta_0$ is equal to α. Furthermore give the operating characteristic (O.C.) curve for this procedure; i.e., plot $L(\theta)$, the probability of accepting a lot having mean life θ, against θ.

Solution: A censored life test involves terminating the test after a preassigned number, r, of failures occur. More precisely, n items are drawn at random from a distribution whose p.d.f. is given by (1) and placed on life test. Observations become available in order; i.e., $x_{1,n} \leq x_{2,n} \leq \cdots \leq x_{r,n} \leq \cdots \leq x_{n,n}$, where by $x_{i,n}$ is meant the time when the i'th failure occurs. Experimentation is terminated as soon as the r'th failure occurs.

In the non-replacement case (where failed items are not replaced), it can be shown that an estimate of θ which is "best" in the sense that it is maximum likelihood, unbiased, minimum variance, efficient and sufficient is given by

$$\text{(2)} \qquad \hat{\theta}_{r,n} = \left[\sum_{i=1}^{r} x_{i,n} + (n-r)\, x_{r,n}\right] \Big/ r .$$

In the replacement case (where one immediately replaces a failed item by a new one) the appropriate "best" estimate is given by

$$\text{(3)} \qquad \hat{\theta}_{r,n} = n x_{r,n} / r,$$

where by $x_{r,n}$ is meant the total time (measured from the beginning of the life test) to observe the r'th failure and where the sample size n is maintained throughout the life test.

The following results are very useful:

(i) The p.d.f. of $\hat{\theta}_{r,n}$ in either the replacement or non-replacement case is given by

(4) $$f_r(y) = \frac{1}{(r-1)!}(r/\theta)^r y^{r-1} e^{-ry/\theta}, \quad y > 0$$

$$= 0, \text{ elsewhere}$$

and further the random variable $W = 2r\hat{\theta}_{r,n}/\theta$ is distributed as chi-square with $2r$ degrees of freedom $(\chi^2(2r))$.

(ii) The expected waiting time for the r'th failure is given by

(5) $$E(X_{r,n}) = \theta \sum_{j=1}^{r} 1/(n-j+1)$$

in the non-replacement case and by

(6) $$E(X_{r,n}) = r\theta/n$$

in the replacement case.

From (4) we can now write down a test procedure having the required property that the probability of rejecting a lot with mean life $\theta = \theta_0$ is equal to α (such a procedure can also be said to have size, type I error, or producer's risk α). The region of acceptance is given by

(7) $$\hat{\theta}_{r,n} > C = \theta_0 \chi^2_{1-\alpha}(2r)/2r,$$

where we define the constant $\chi^2_\gamma(2k)$ by the equation

(8) $$\Pr(\chi^2(2k) > \chi^2_\gamma(2k)) = \gamma.$$

The O.C. curve associated with this procedure is given by

(9) $$L(\theta) = \text{Prob}(\hat{\theta}_{r,n} > \theta_0 \chi^2_{1-\alpha}(2r)/2r \mid \theta)$$

$$= \text{Prob}(\chi^2(2r) > \frac{\theta_0}{\theta}\chi^2_{1-\alpha}(2r)).$$

It is convenient to choose units in such a way that $\theta_0 = 1$. If this is done, (7) and (9) become, respectively, (7') $$\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$$

and

(9') $$L(\theta) = \Pr\left(\chi^2(2r) > \chi^2_{1-\alpha}(2r)/\theta\right).$$

In Table 1 we give the values of $\chi^2_{1-\alpha}(2r)/2r$ for $r = 1(1)10(5)30(10)50(25)100$ for $\alpha = .01, .05, .10, .25$, and $.50$. The untabulated values corresponding to $r \leq 100$ can be found from tables of chi-square. For values of $r > 100$, the normal approximation to chi-square, or the more refined approximations due to Fisher or to Wilson and Hilferty can be used.

In Tables 2 (a), (b), (c), (d), (e) we give the values of θ which are accepted with probability $p = .99, .95, .90, .75, .50, .25, .10, .05$, and $.01$ if a lot with $\theta = 1$ is accepted with probability .99, .95, .90, .75 and .50, respectively. O. C. curves based on Tables 2 (a), (b), (c), (d), and (e) are given for the r values noted above in Figures 1 (a) through 1 (e).

All of the test procedures given by (7) have the property that $L(\theta_0) = 1-\alpha$ independently of n. Changes in n affect $E(X_{r,n})$, the expected waiting time to observe the r'th value in a sample of size n. The appropriate choice of n for a given r and hence fixed type I error, α, depends on economic considerations and involves balancing the cost of increasing n, with the gains due to decreasing the expected waiting time of the experiment. To facilitate the making of such judgments we give values of $E(X_{r,n})$ for $r = 1(1)n$ and $n = 1(1)20(5)30(10)100$ in Table 3 (a) and for $r = 1(1)10(5)30(10)50(25)100$ and $n = kr$, with $k = 1(1)10(10)20$ in Table 3 (b).

It is particularly interesting to compare the following two procedures: (1) a test based on $\hat{\theta}_{r,r}$ where r items are placed on test and where one waits for all r items to fail and (2) a test based on $\hat{\theta}_{r,n}$ where n items are placed on test and where one waits only for the first r

failures to occur. From (5) the expected waiting times in the non-replacement case are given by

$$E(X_{r,r}) = \theta \sum_{j=1}^{r} 1/j \quad \text{and} \quad E(X_{r,n}) = \theta \sum_{j=1}^{r} 1/n-j+1 \text{, respectively.}$$

Thus the ratio $\alpha_{r,n} = E(X_{r,n})/E(X_{r,r})$ is a measure of the expected saving in time due to using the second procedure rather than the first procedure. Values of $\alpha_{r,n}$ can be computed readily from Tables 3 (a) and 3 (b). A brief table is given in Table 3 (c). It follows from (6) that the expected waiting times in the replacement case are given by $E(X_{r,r}) = \theta$ and $E(X_{r,n}) = r\theta/n$. In this case the ratio $\alpha_{r,n}$ is simply $\alpha_{r,n} = E(X_{r,n})/E(X_{r,r}) = \frac{r}{n}$.

Numerical Example: Let us compare the average length of time needed to observe (a) the failure of the first 2 out of 4 items under test with the average length of time required to observe, (b) the failure of 2 out of 2 items. The answer is $\alpha_{2,4} = E(X_{2,4})/E(X_{2,2}) = \frac{7}{12} / \frac{3}{2} = 7/18 = .3889$.

Hence it will take on the average only 7/18 as long to observe the first 2 out of 4 items as 2 out of 2. If life is exponential we know that tests based on either (a) or (b) have the same O. C. curves; however, the time required for (a) is on the average substantially shorter than for (b) [average time for (a) is about 40% of the average time required for (b)].

Remark: It was noted above that for given type I error (or producer's risk) α and stopping number r, all test procedures (7) have the same O. C. curve independent of n. We wish to give a method for choosing a best procedure from this class of procedures. It is clear that for fixed α and r, increasing n will on the one hand cut the expected waiting time, but will, on the other hand, increase the cost due to placing more items on test. More precisely, if c_1 is the cost of waiting per hour and c_2 is the cost of placing an item on test, then the total expected cost associated with a plan based on (7), assuming that the mean life is close to Θ_o, is given by $c_1 \Theta_0(\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{n-r+1}) + c_2 n$ in the non-replacement case and $c_1 \Theta_o \frac{r}{n} + c_2 n$ in the replacement case. It is clear that there exists an n which makes this quantity a minimum and that this minimum depends only on the ratio c_2/c_1. Table 3 (a) is useful in reaching a conclusion as to the best n.

To illustrate the point in the non-replacement situation consider a case where $r = 10$, $\Theta_o = 1000$ hrs, $c_1 = \$1$ per hour, $c_2 = \$100$ per item tested. Then we have the following table:

n	Expected cost due to waiting	Cost of items tested	Total cost
10	2929	1000	3929
11	2020	1100	3120
12	1603	1200	2803
13	1346	1300	2646
14	1168	1400	2568
15	1035	1500	2535
16	931	1600	2531
17	847	1700	2547

The minimum is attained for $n = 16$.

It is easily verified that for the given values of r, Θ_o, c_1, and c_2, the optimum n is 10 in the replacement case.

We have just given a numerical example. It is of interest to give a general method for finding the optimum n. In the replacement case the optimum n is the integer nearest to $\sqrt{\frac{c_1 \Theta_o r}{c_2} + \frac{1}{4}}$ (or either m or $m + 1$ if $\sqrt{\frac{c_1 \Theta_o r}{c_2} + \frac{1}{4}} = \frac{2m+1}{2}$ for some integer m).

In the non-replacement case, we choose the smallest n , such that $E(X_{r,n}) - E(X_{r,n+1}) < \frac{c_2}{c_1 \theta_o}$. In the above numerical example, $c_2/c_1 \theta_o = .1$ and $E(X_{10,n}) - E(X_{10,n+1}) > .1$ for $n = 10, 11, 12, 13, 14, 15$, and $E(X_{10,n}) - E(X_{10,n+1}) < .1$ for $n \geq 16$. Therefore, the appropriate $n = 16$.

The procedures described by (7) depend on knowing the first r values $x_1 \leq x_2 \leq \cdots \leq x_r$. It is interesting that if the underlying distribution is exponential, then one can find a truncated procedure having almost precisely the same O.C. curve as the $\hat{\theta}_{r,n}$ procedure. The only requirement that this be so is that n be moderately larger than r. The advantage of such procedures is that they depend only on an extremely simple observation, the time of failure of the r'th item. Procedures of the form, accept if $x_{r,n} > T$ and reject otherwise, are very simple to interpret. In words, the rule of action is to stop experimentation at $\min \cdot [x_{r,n};T]$ with acceptance of the hypothesis if $\min [x_{r,n};T] = T$ (since in that case the r'th failure occurs after time T) and with rejection of the hypothesis if $\min(x_{r,n};T) = x_{r,n}$ (since in that case the r'th failure occurs prior to time T). Experimentation is actually truncated at T and at $x_{r,n}$, respectively, in the two situations.

To derive the truncated procedures one proceeds as follows:

In the replacement case (7) becomes

(10) $$x_{r,n} > \theta_o \; \chi^2_{1-\alpha}(2r)/2n$$

and (7^1) becomes (when θ_o is normalized as 1) ,

(10^1) $$x_{r,n} > \chi^2_{1-\alpha}(2r)/2n.$$

In the non-replacement case, the exact procedure is obtained by first

In the non-replacement case, we choose the smallest n, such that $E(X_{r,n}) - E(X_{r,n+1}) < \frac{c_2}{c_1 \theta_o}$. In the above numerical example, $c_2/c_1\theta_o = .1$ and $E(X_{10,n}) - E(X_{10,n+1}) > .1$ for $n = 10, 11, 12, 13, 14, 15$, and $E(X_{10,n}) - E(X_{10,n+1}) < .1$ for $n \geq 16$. Therefore, the appropriate $n = 16$.

The procedures described by (7) depend on knowing the first r values $x_1 \leq x_2 \leq \ldots \leq x_r$. It is interesting that if the underlying distribution is exponential, then one can find a truncated procedure having almost precisely the same O.C. curve as the $\hat{\theta}_{r,n}$ procedure. The only requirement that this be so is that n be moderately larger than r. The advantage of such procedures is that they depend only on an extremely simple observation, the time of failure of the r'th item. Procedures of the form, accept if $x_{r,n} > T$ and reject otherwise, are very simple to interpret. In words, the rule of action is to stop experimentation at $\min [x_{r,n};T]$ with acceptance of the hypothesis if $\min [x_{r,n};T] = T$ (since in that case the r'th failure occurs __after__ time T) and with rejection of the hypothesis if $\min(x_{r,n};T) = x_{r,n}$ (since in that case the r'th failure occurs prior to time T). Experimentation is actually truncated at T and at $x_{r,n}$, respectively, in the two situations.

To derive the truncated procedures one proceeds as follows:

In the __replacement__ case (7) becomes

$$(10) \qquad x_{r,n} > \theta_o \chi^2_{1-\alpha}(2r)/2n$$

and (7^1) becomes (when θ_o is normalized as 1),

$$(10^1) \qquad x_{r,n} > \chi^2_{1-\alpha}(2r)/2n .$$

In the __non-replacement__ case, the __exact__ procedure is obtained by first writing down the distribution of the r'th smallest value in a sample of size n. In the case where $\Theta_0 = 1$, the p.d.f. of $x_{r,n}$ is given by

(11) $$g_{r,n}(x) = \frac{n!}{(r-1)!(n-r)!} [1-e^{-x}]^{r-1} e^{-(n-r+1)x}, \; x > 0.$$

Therefore, $\Pr(X_{r,n} > T)$ becomes

(12) $$\Pr(X_{r,n} > T) = \int_T^\infty g_{r,n}(x)dx = \int_T^\infty \frac{n!}{(r-1)!(n-r)!} [1-e^{-x}]^{r-1} e^{-(n-r+1)x} dx \; .$$

Letting $u = 1-e^{-x}$, we get

(13) $$\Pr(X_{r,n} > T) = n\binom{n-1}{r-1} \int_{(1-e^{-T})}^1 u^{r-1}(1-u)^{n-r} du \; .$$

If we want to solve for T in the equation $\Pr(X_{r,n} > T) = 1-\alpha$, we can do this readily from tables of the incomplete Beta distribution. An alternative procedure is to use tables of the cumulative binomial distribution. If one defines $B(k;n,p)$ as

(14) $$B(k;n,p) = \sum_{\nu=0}^{k} b(\nu;n,p), \quad \text{where}$$

$b(\nu;n,p) = \binom{n}{\nu} p^\nu(1-p)^{n-\nu}$, then one can show that (13) becomes

(15) $$1-B(r-1;n,p) = n\binom{n-1}{r-1} \int_0^p u^{r-1}(1-u)^{n-r}du,$$

where $p = 1-e^{-T}$. To solve the problem at hand, the procedure is to compute p so that

(16) $$1-B(r-1;n,p) = \sum_{k=r}^{n} \binom{n}{k} p^k(1-p)^{n-k} = \alpha$$

and then set $T = \log(\frac{1}{1-p})$.

An alternative <u>approximate</u> procedure in the non-replacement case which has been shown to be extremely close to the exact procedure is obtained as follows:

In (7) or (7') replace the unbiased estimator $\hat{\theta}_{r,n}$ by $\beta_{r,n}\, x_{r,n}$ where $\beta_{r,n} = 1/E(X_{r,n})$. It has been observed that if n is sufficiently larger than r, then the O.C. curve resulting from using the acceptance region $\beta_{r,n}\, x_{r,n} > C$ virtually coincides with the O.C. curve associated with $\hat{\theta}_{r,n} > C$. Consequently a test procedure having virtually the same O.C. curve as (7) or (7') is given by

(17) $$x_{r,n} > T = \theta_o \chi^2_{1-\alpha}(2r)/2r\,\beta_{r,n} = \theta_o \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$$

or

(17') $$x_{r,n} > \chi^2_{1-\alpha}(2r)/2r\,\beta_{r,n} = \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$$

in the normalized case $\theta_o = 1$.

In Table 4 we give values of $\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ for the values of r and α covered in Table 1 and for various values of n. If r and n are sufficiently large, then $\frac{1}{\beta_{r,n}} \sim \log(\frac{n}{n-r})$. This approximation is useful for computing some values of T when r and n lie outside Table 4.

<u>Remark:</u> An approximate formula for p_α satisfying $\sum_{k=r}^{n} \binom{n}{k} p_\alpha^{\,k}(1-p_\alpha)^{n-k} = \alpha$ is found by solving for p_α in the equation:

$\log(\frac{1}{1-p_\alpha}) = \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$. Solving for p_α one gets:

$$p_\alpha = 1-e^{-\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r}.$$

<u>Example:</u> $r = 10$, $n = 20$. Find p_α such that $\sum_{k=10}^{20} \binom{20}{k} p_\alpha^{\,k}(1-p_\alpha)^{20-k} = \alpha$ for $\alpha = .01, .05, .10, .25, .50$. In column 2 we give the approximate

value of p_α, as obtained by the formula, and in column 3 we give the value of p_α computed by interpolation from the Harvard tables of the Cumulative Binomial Distribution.

α	p_α (approximate)	p_α (exact)
.01	.241	.239
.05	.305	.302
.10	.340	.338
.25	.404	.402
.50	.476	.475

Numerical Examples:

1. Find a censored life test which will accept a lot having a mean life of 1000 hours with probability .90. The experiment is to be stopped after one has observed the first 5 failures.

Solution: In terms of the notation that we have used $\theta_0 = 1000$, $\alpha = .10$, and $r = 5$. In the non-replacement case $\hat{\theta}_{5,n} = [x_1+x_2+x_3+x_4+(n-4)x_5]/5$, and in the replacement case $\hat{\theta}_{5,n} = nx_5/5$. The region of acceptance is given by $\hat{\theta}_{5,n} > \theta_0\chi^2_{1-\alpha}(2r)/2r = (1000)\,\chi^2_{.9}\,(10)/10$

$$= 486.5.$$

In words, place n items on test. Wait until the first 5 failures occur. Compute $\hat{\theta}_{5,n}$. Accept the hypothesis that $\theta_0 = 1000$ if $\hat{\theta}_{5,n} > 486.5$, reject otherwise.

2. If the above procedure is used, find the probability of accepting a lot having mean life $\theta = 500$, $\theta = 250$.

Solution: The result can be obtained readily from the O.C. curve in Figure 1. Analytically we use formula (9). From this we get

$$L(500) = Pr\left\{\chi^2(10) > 2\chi^2_{.9}(10)\right\} = Pr\left\{\chi^2(10) > 9.730\right\}$$
$$= .47$$

$$L(250) = Pr\left\{\chi^2(10) > 4\chi^2_{.9}(10)\right\} = Pr\left\{\chi^2(10) > 19.46\right\}$$
$$= .037.$$

3. If the above procedure is used, for what value of Θ is $L(\Theta)$, the probability of acceptance = .50? = .10? = .05?

__Solution:__ Using table 2 (c), $L(\Theta) = .50$, for $\Theta = 521$; $L(\Theta) = .10$, for $\Theta = 304$; and $L(\Theta) = .05$, for $\Theta = 266$.

4. Find the expected waiting time in the non-replacement case for the following choices of n: 5, 10, 20, assuming $\Theta = \Theta_0 = 1000$.

__Solution__: From formula 5 or more easily from Table 3 (a) or 3 (b) we get

$$E(X_{5,5}) = 2283, \quad E(X_{5,10}) = 645.6, \quad \text{and} \quad E(X_{5,20}) = 279.5.$$

5. Same as 4 in the replacement case.

__Solution:__ $E(X_{r,n}) = r\Theta_0/n$

and $E(X_{5,5}) = 1000$, $E(X_{5,10}) = 500$ and $E(X_{5,20}) = 250$.

6. Find a truncated procedure based on $X_{5,10}$ in the non-replacement case, with Type I error $\alpha = .10$.

__Exact solution:__ Compute p so that

$$\sum_{k=5}^{10} \binom{10}{k} p^k(1-p)^{10-k} = .10 .$$

Using binomial tables we find $p = .267$. Thus

$$T = \log \frac{1}{1-p} = \log \frac{1}{.733} = .3106.$$

Thus the test procedure is: Accept if $x_{5,10} > 311$ and reject otherwise.

According to (17) an excellent approximation is given by letting

$$T = \theta_o \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r = (486.5)(.6456) = 314 .$$

See also table 4 (c) with $r = 5$ and $n = 2r = 10$.

The two values: 311 (exact) and 314 (approximate) are very close.

7. Find a truncated procedure based on $x_{5,10}$ in the replacement case.

Solution: According to (10) the appropriate truncation procedure is to accept if

$$x_{5,10} > (1000)(4.865)/20 = 243.2$$

and reject otherwise.

Section 2

A censored test having the property that its O.C. curve is such that

$$L(\theta_o) = 1 - \alpha \quad \text{and} \quad L(\theta_1) \leq \beta .$$

One frequently wishes to design a life test which requires that the O.C. curve meet the following prescribed conditions:

(i) If $\theta = \theta_o$, then $L(\theta_o) = 1 - \alpha$

(ii) If $\theta = \theta_1$, then $L(\theta_1) \leq \beta$

where $\theta_o > \theta_1$.

Put into words we have a situation where lots having mean life $\theta \geq \theta_o$ are considered desirable; lots having mean life $\theta \leq \theta_1$ are considered undesirable. The interval (θ_1, θ_o) is essentially a zone of indifference. The α and β can be thought of as producer's and consumer's risks or as errors of the first and second kind, respectively.

The problem amounts to choosing r in (7) in such a way that not only is $L(\theta_o) = 1 - \alpha$, but also that $L(\theta_1) \leq \beta$. From (7) and (9) it is clear that these two conditions are met if r is such that

$$(18) \qquad \frac{\theta_o}{\theta_1} \chi^2_{1-\alpha}(2r) \geq \chi^2_{\beta}(2r) \quad \text{or} \quad \frac{\theta_1}{\theta_o} \leq \chi^2_{1-\alpha}(2r)/\chi^2_{\beta}(2r) .$$

More precisely we want the <u>smallest</u> r meeting this condition; i.e., we want that integer r which is such that the associated O.C. curve passes most nearly through the points $[\theta_o , L(\theta_o) = 1 - \alpha]$ and $[\theta_1 , L(\theta_1) = \beta]$. It is readily verified that as r goes through the values 1, 2, 3, ... the ratio $\chi^2_{1-\alpha}(2r)/\chi^2_{\beta}(2r)$ increases monotonically to unity. Consequently we can always find an integer r such that

$$(19) \qquad \chi^2_{1-\alpha}(2r)/\chi^2_{\beta}(2r) \geq \frac{\theta_1}{\theta_o} > \chi^2_{1-\alpha}(2r-2)/\chi^2_{\beta}(2r-2) .$$

This is the value of r which we want to use. Using this value of r, the region of acceptance

$$\hat{\theta}_{r,n} > \theta_o \chi^2_{1-\alpha}(2r)/2r \tag{20}$$

is such that its associated O.C. curve has $L(\theta_o) = 1-\alpha$ and $L(\theta_1) \leq \beta$.

In Table 5 we give the appropriate values of r and of $\chi^2_{1-\alpha}(2r)/2r$ for the 16 number pairs (α, β) which can be made with the numbers (.01, .05, .10, .25) and the values $k = \frac{\theta_o}{\theta_1} = \frac{3}{2}$, 2, 3, 5, and 10.

Remark: It will be noted that the values of r required for $k = \frac{3}{2}$ are quite large. It is our feeling that, generally speaking, it is rare that one would want to work with values of $k < \frac{3}{2}$. In case this is so, however, we should like to indicate how we would find the required r and $\chi^2_{1-\alpha}(2r)/2r$.

Since r must be large we use the approximation that $\chi^2(2r)$ is distributed approximately normally with mean $2r$ and standard deviation $2\sqrt{r}$. Thus to require that the O.C. curve be such that

$$L(\theta_o) = 1-\alpha \text{ and } L(\theta_1) \leq \beta \text{ with } \frac{\theta_o}{\theta_1} = k < \frac{3}{2},$$

we choose the smallest integer r such that

$$\frac{\chi^2_{\beta}(2r)}{\chi^2_{1-\alpha}(2r)} \leq \frac{\theta_o}{\theta_1} = k.$$ This means, using the normal approximation, finding r such that

$$\frac{2r + 2C_{\beta}\sqrt{r}}{2r + 2C_{\alpha}\sqrt{r}} = k \quad \text{or} \quad r = \left(\frac{C_{\beta} + k\,C_{\alpha}}{k-1}\right)^2 .$$

C_{α} or C_{β} = 2.326, 1.645, 1.282, .674 for α or β = .01, .05, .10, .25, respectively.

Once we have found r, then $\chi^2_{1-\alpha}(2r)/2r$ is given by

$$\frac{2r - 2C_\alpha \sqrt{r}}{2r} = 1 - \frac{C_\alpha}{\sqrt{r}} .$$

Numerical Examples

1. Design a censored life test which will meet the following conditions: When $\theta_o = 1500$, $L(\theta_o) = .95$ and when $\theta_1 = 500$, $L(\theta_1) \leq .05$.

<u>Solution:</u> In the problem $k = \theta_o/\theta_1 = 3$, $\alpha = .05$, and $\beta = .05$. Using Table 5, we see that $r = 10$ and the region of acceptance is given by

$$\hat{\theta}_{10,n} > C = \theta_o \chi^2_{.95}(20)/20 = 815.$$

For this procedure $L(\theta_o) = .95$ exactly and $L(\theta_1) = .038$.

2. Find the appropriate r for the case where $k = \theta_o/\theta_1 = 1.1$ and $\alpha = \beta = .05$.

<u>Solution:</u> Using the remark, we know that

$$r = \left(\frac{1.645 + (1.1)(1.645)}{.1}\right)^2 = \left[(21)(1.645)\right]^2$$
$$= (34.55)^2 = 1194.$$

Truncated life tests having the property that the associated O.C. curve is such that $L(\theta_o) \geq 1-\alpha$ and $L(\theta_1) \leq \beta$.

In the previous section we considered censored life tests, i.e., tests in which life testing stops after a prescribed number of failures, r, have occured. While such tests do in general have the desirable effect of shortening experiment time, there is nevertheless the ever present feature that one does not know precisely when the experiment will end, since this depends on the random time $X_{r,n}$. As a matter of fact, it is frequently necessary because of practical considerations to terminate a life test by a preassigned time T_o, a requirement which censored tests do not meet.

If we wish to truncate an experiment by a preassigned time T_o we are led to truncated life tests in which it is decided in advance that the life test will be terminated at $\min(X_{r_o,n}; T_o)$ where $X_{r_o,n}$ is the time when the r_o'th failure occurs and T_o is the truncation time beyond which the life test will not be allowed to run. (Both r_o and T_o are preassigned.) If the life test is terminated at $X_{r_o,n}$ (i.e., r_o failures occur before time T_o), then the action taken will be to reject. If the experiment is terminated at time T_o (i.e., the r_o'th failure occurs after T_o), then the action in terms of hypothesis testing is acceptance. It can be shown that three functions characterize the test procedures in either the replacement or non-replacement case. These are:

(i) $E_\Theta(r)$, the expected number of items failing before reaching a decision,

(ii) $E_\Theta(T)$, the expected waiting time to reach a decision, and

(iii) $L(\Theta)$, the probability of accepting, if the true value of the mean life is Θ.

In the non-replacement case

$$E_\Theta(r) = np_\Theta B(r_o - 2; n - 1, p_\Theta) + r_o[1 - B(r_o - 1; n, p_\Theta)] \tag{21}$$

where $p_\Theta = 1 - e^{-T_o/\Theta}$.

The probability distribution of r is given by

(22) $$\Pr(r=k|\theta) = b(k;n,p_\theta),\ k = 0,1,2,\ldots,r_o-1$$

and

(22') $$\Pr(r=r_o|\theta) = 1 - B(r_o-1;n,p_\theta) .$$

Further, one has

(23) $$E_\theta(T) = \sum_{k=1}^{r_o} \Pr(r=k|\theta)\, E_\theta(X_{k,n}) ,$$

where $E_\theta(X_{k,n})$ can be found from (5), and

(24) $$L(\theta) = \sum_{k=0}^{r_o-1} \Pr(r=k|\theta) = B(r_o-1;n,p_\theta) .$$

In the replacement case the probability distribution of r is given by

(25) $$\Pr(r=k|\theta) = p(k;\lambda_\theta),\ k = 0,1,2,\ldots,r_o-1$$

(25') $$\Pr(r=r_o|\theta) = 1 - \sum_{k=0}^{r_o-1} p(k;\lambda_\theta) = 1 - \pi(r_o-1;\lambda_\theta)$$

In (25) and (25'), $\lambda_\theta = nT_o/\theta$, $p(k;\lambda_\theta) = e^{-\lambda_\theta}\lambda_\theta^k/k!$

and

$$\pi(r;\lambda_\theta) = \sum_{k=0}^{r} p(k;\lambda_\theta) .$$

Further, one has

(26) $$E_\theta(r) = \lambda_\theta\pi(r_o-2;\ \lambda_\theta) + r_o[1-\pi(r_o-1;\lambda_\theta)]$$

(27) $$E_\theta(T) = \theta E_\theta(r)/n$$

and

(28) $$L(\theta) = \pi(r_o-1;\lambda_\theta) .$$

We have just given formulae for the O.C. curve, the expected waiting time, and expected number of items failed in the course of reaching a decision for any preassigned n, T_o, r_o. The problem is to find the appropriate truncated test (i.e., to find r_o and n) when the truncation time T_o is preassigned and the O.C. curve is required (for preassigned type I error α and type II error β) to be such that $L(\Theta_o) \geq 1-\alpha$ and $L(\Theta_1) \leq \beta$. It can be shown that for both the replacement and non-replacement cases the appropriate r_o is precisely the one used in the censored test (20) and tabulated in Table 5. In the replacement case, the appropriate value of n one should choose is given by

$$n = \left[\Theta_o \chi^2_{1-\alpha}(2r_o)/2T_o\right] \tag{29}$$

where $[x]$ means the greatest integer $\leq x$.

In the non-replacement situation a good approximate value of n, in case Θ_o/T_o is substantially more than one (say ≥ 3), is given by

$$n = \left[r_o/(1-e^{-T_o/C})\right] \tag{30}$$

where $C = \Theta_o \chi^2_{1-\alpha}(2r_o)/2r_o$ and where r_o is the same as in the replacement case.

In Table 6 (7) we give the appropriate values of n to use in the replacement (non-replacement) case when $\alpha = .01, .05, .10, .25$; $\beta = .01, .05, .10, .25$; $\Theta_o/\Theta_1 = \frac{3}{2}, 2, 3, 5, 10$; and $\Theta_o/T_o = 3, 5, 10, 20$. The values of n have been checked by computing $L(\Theta_o)$ and $L(\Theta_1)$; the O.C. curve of the truncated test does come very close to meeting the requirements $L(\Theta_o) \geq 1-\alpha$ and $L(\Theta_1) \leq \beta$.

Several remarks appear to be relevant at this point and should be made:

Remark 1: In the replacement case the O.C. curve of the test based on $\min(X_{r_{o,n}}; T_o)$, where the values of r_o are given in Table 5 and where n is given by (29), is such that $L(\theta_o) \geq 1-\alpha$, but in some cases it may happen that $L(\theta_1)$ may be slightly $> \beta$. This can be avoided in either of two ways. One way is to give the experimenter the freedom to use, instead of the truncation time T_o, the slightly larger truncation time $T_o' = \theta_o \chi^2_{1-\alpha}(2r_o)/2n$. The test based on $\min(X_{r_{o,n}}; T_o')$ will have $L(\theta_o) = 1-\alpha$ and $L(\theta_1) \leq \beta$. The other way is to use $(n+1)$ items throughout the test and to use, instead of T_o, the slightly smaller truncation time $T_o'' = \theta_o \chi^2_{1-\alpha}(2r_o)/2(n+1)$. The test based on $\min(X_{r_{o,n+1}}; T_o'')$ will have $L(\theta_o) = 1-\alpha$ and $L(\theta_1) \leq \beta$. In most cases it will be a matter of indifference which procedure one adopts.

Remark 2: A good approximate solution for finding a truncated non-replacement test procedure was given by (30). An alternative, more direct (and also more lengthy) procedure for finding a truncated non-replacement test meeting the conditions prescribed is to note that such a test is equivalent to a binomial situation in which we test $p_o = 1-e^{-T_o/\theta_o}$ against $p_1 = 1-e^{-T_o/\theta_1}$ and want the O.C. curve to be such that $L(p_o) \geq 1-\alpha$ and $L(p_1) \leq \beta$. Stated in the language of sampling inspection, we are seeking a sample size n and a rejection number r_o such that the resulting O.C. curve has the property that $L(p) \geq 1-\alpha$ for lots with $p \leq p_o$ and $L(p) \leq \beta$ for lots with $p \geq p_1$. The detailed calculations necessary to determine n and r_o can be carried out in any given situation by using the Binomial Tables or Tables of the Incomplete Beta Function.

Remark 3: It is appropriate to mention that truncated test procedures of the kind considered in this section are good rules of action in cases where the underlying life distribution is not necessarily exponential. More precisely, we mean the following: Suppose that an acceptable lot of electron tubes is one for which the probability of failing before some time T_o is $\leq p_o$ and that an unacceptable lot is one for which the probability of failure before some time T_o is $\geq p_1 (p_1 > p_o)$ and suppose we want to O.C. curve to be such that $L(p_o) \geq 1-\alpha$ and $L(p_1) \leq \beta$. It is clear that the comments made in remark 2 are relevant here and that the test procedure involves finding a sample size n and rejection number r_o such that we will accept the hypothesis that $p = p_o$ if the number of defectives (failures before T_o) in the sample $\leq (r_o - 1)$ and reject the hypothesis that $p = p_o$ (accept $p = p_1$) if the number of defectives in the sample $\geq r_o$. This test procedure clearly is truncated and has the property that $L(p_o) \geq 1-\alpha$ for <u>any</u> distribution $F_o(x)$ which is such that $\int_o^{T_o} dF_o(x) \leq p_o$ and $L(p_1) \leq \beta$ for any distribution $F_1(x)$ which is such that $\int_o^{T_o} dF_1(x) \geq p_1$. If, in particular, $F_o(x) = 1-e^{-x/\theta_o}$, with $\theta_o = T_o/\log \frac{1}{1-p_o}$ and $F_1(x) = 1-e^{-x/\theta_1}$, with $\theta_1 = T_o/\log \frac{1}{1-p_1}$ the test procedure just described has the property that $L(\theta_o) \geq 1-\alpha$ and $L(\theta_1) \leq \beta$. Recalling that the rule of action can be written as <u>accept</u> if $\min(X_{r_o n};T_o) = T_o$ and <u>reject</u> if $\min(X_{r_o n};T_o) = X_{r_o,n}$, we have precisely the truncated procedure which one gets in the exponential case when testing θ_o against θ_1 with $L(\theta_o) \geq 1-\alpha$ and $L(\theta_1) \leq \beta$. But from the preceding argument the test procedure is distribution free in the sense that it is the appropriate one to use when we wish to distinguish between two distributions $F_o(x)$ and $F_1(x)$ with

$$\int_o^{T_o} dF_o(x) \leq p_o = 1-e^{-T_o/\theta_o} \quad \text{and} \quad \int_o^{T_o} dF_1(x) \geq p_1 = 1-e^{-T_o/\theta_1} \quad .$$

For all such cases $L(F_o) \geq 1-\alpha$ and $L(F_1) \leq \beta$. In this connection it is useful to point out how Table 8 can be used to design single sample plans which are such that when $p = p_o$, the probability of acceptance equals $1-\alpha$

and when $p = p_1$, the probability of rejection is greater than or equal to $1 - \beta$. It is assumed that $0 \leq p_o \leq p_1 \leq .10$ and that the producers' risk α and consumers' risk β are allowed to assume the three values .01, .05, .10, thus giving rise to nine possible pairs (α , β) . In entering the table, the ratio p_1/p_o should be identified with θ_o/θ_1 . When this is done the rejection number is given by r_o and the sample size is given by $\left[\chi^2_{1-\alpha}(2r_o)/2p_o\right]$ where $[x]$ means the greatest integer $\leq x$. In this table, r_o is the upper number and $\chi^2_{1-\alpha}(2r_o)/2$ is the lower number.

The justification for the procedure stems from the fact that the probability of drawing a defective item from a lot having fraction defective p is, for small p , essentially the same as the probability that a failure occurs when one observes a Poisson process having failure rate p (per unit time) or mean life $1/p$, for a unit of observation time. Thus, drawing a sample of size N from a lot having fraction defective p can be thought of as drawing one item at random from a lot whose items follow a life distribution which is exponential with mean life $\theta = 1/p$, placing this item under test; replacing it, when it fails, by a new item drawn form the lot, and terminating experimentation either as soon as r_o failures occur or when $T_o = N$ units of time have elapsed, whichever comes first. This means placing $n = 1$, $\theta_o = 1/p_o$ in formula (29) and getting the sample size $N = T_o = \left[\chi^2_{1-\alpha}(2r_o)/2p_o\right]$.

Numerical Examples

1. Find a truncated replacement plan for which $T_o = 500$ hours, which will accept a lot with mean life = 10,000 hours at least 95 per cent of the time and reject a lot with mean life = 2000 hours at least 95 per cent of the time. Compute $L(\theta), E_\theta(T)$, and $E_\theta(r)$ at $\theta = 10{,}000$ and $\theta = 2000$, respectively.

Solution: In this case $\theta_o = 10{,}000$, $\theta_1 = 2{,}000$, $\alpha = \beta = .05$.

Since $\theta_o/\theta_1 = 5$, it follows from Table 5 that $r_o = 5$. From Table 6, we find that corresponding to $\theta_o/\theta_1 = 5$, $\theta_o/T_o = 20$, $\alpha = \beta$ .05, $n = 39$. Thus the following truncated replacement plan meets the requirements: Start the life test with $n = 39$ items. As soon as one item fails, replace it by a new item. Accept the lot if $\min(X_{5,39}; 500) = 500$ and reject the lot if $\min(X_{5,39}; 500) = X_{5,39}$. If the lot is rejected, experimentation is stopped at $X_{5,39}$, the time of occurrence of the fifth failure.

For $\theta = 10{,}000$, $\lambda_\theta = nT_o/\theta = (39)(500)/10{,}000 = 1.95$. Using Molina's Tables, one finds from (28) that $L(\theta) = .952$. Substituting in (26) and (27), respectively, gives $E_\theta(r) = 1.93$ and $E_\theta(T) = 495$. For $\theta = 2{,}000$, $\lambda_\theta = nT_o/\theta = (39)(500)/2{,}000 = 9.75$. For this value of θ, $L(\theta) = .034$, $E_\theta(r) = 4.95$, and $E_\theta(T) = 254$.

2. Same as 1 except that we want a non-replacement procedure.

Solution: $r_o = 5$. According to Table 7, the sample size is $n = 42$. For $\theta = 10{,}000$, $T_o/\theta = .05$, and $p_o = 1-e^{-T_o/\theta} = .049$. Using the Binomial Tables one finds from (24) that $L(\theta) = .946$. Substituting in (21) and (23) gives $E_\theta(r) = 2.02$ and $E_\theta(T) = 494$. For $\theta = 2{,}000$, $T_o/\theta = .25$. For this value of θ, $L(\theta) = .031$, $E_\theta(r) = 4.91$, and $E_\theta(T) = 248$.

3. Consider the truncated replacement plan meeting the conditions of Problem 1.

For what values of θ is $L(\theta) = .5$? What are $E_\theta(r)$ and $E_\theta(T)$ for this value of θ ?

Solution: To find the θ such that $L(\theta) = .5$ means finding λ_θ such that $\pi(4;\lambda_\theta) = .5$. Using the Molina tables, we see that this means $\lambda_\theta = 4.67$. Since $\lambda_\theta = nT_o/\theta$, with $n = 39$, $T_o = 500$, one finds $\theta = 4180$. From (26) and (27) we find $E_\theta(r) = 3.97$ and $E_\theta(T) = 424$.

4. Consider the truncated non-replacement plan meeting the conditions of Problem 2 . For what values of θ is $L(\theta) = .5$?

Solution: This means finding p_θ such that $B(4;42,p_\theta) = .5$. Using the Binomial tables this means $p_\theta = .1104$. Since $p_\theta = 1-e^{-T_o/\theta}$, the appropriate $\theta = 4274$. In this problem $E_\theta(r)$ and $E_\theta(T)$ will be approximately the same as in the replacement case. The exact calculations can be made from (21) , (22), and (23) and are left to the reader.

5. Find a life test having the following properties: It will accept at least 95% of the lots for which the probability of failing before some time T_o is $\leq .01$ and will reject at least 90% of the lots for which the probability of failing before T_o is $\geq .05$.

Solution: In line with what was said in Remark 3 this means finding the appropriate sampling plan for the special case where $p_o = .01$, $p_1 = .05$, $\alpha = .05$, $\beta = .10$. In this case $p_1/p_o = 5, r_o = 4$, and from Table 8, the sample size $N = (1.37)/.01 = 137$. Thus the life test is as follows: Place 137 items on test. If 4 or more failures occur before time T_o , reject. If 3 or fewer failures occur before time T_o , accept. It should be noted that we are not making any assumption about the underlying distribution of life.

Section 4

Sequential Life Tests

It can be shown that sequential life tests are superior to either censored or truncated life tests. It is shown in a paper by Epstein and Sobel that the sequential probability ratio test of A.Wald can be applied to life testing. The interesting point now is that decisions can be made continuously in time. At each moment t, one can decide either to accept, to reject, or to continue the life test. If we are, as before, testing $H_o: \theta = \theta_o$ against $H_1 : \theta = \theta_1$ $(\theta_o > \theta_1)$ with Type I error $= \alpha$ and Type II error $= \beta$, then the decision as time unfolds depends on

$$(31) \qquad B < \left(\frac{\theta_o}{\theta_1}\right)^r \exp\left[-\left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(t)\right] < A ,$$

where A and B can for all practical purposes be taken as

$$(32) \qquad A = (1-\beta)/\alpha \quad \text{and} \quad B = \beta/(1-\alpha) \quad .$$

In (31) r is the number of failures observed by time t and $V(t)$ is a statistic which equals the total number of hours lived by all items, failed and unfailed, up to time t. In the replacement case

$$(33) \qquad V(t) = nt \quad ,$$

while in the non-replacement case

$$(34) \qquad V(t) = \sum_{i=1}^{r} (n-i+1)(x_i - x_{i-1}) + (n-r)(t-x_r)$$

$$= \sum_{i=1}^{r} x_i + (n-r)t \quad .$$

In (34) x_i denotes the time of the i'th failure.

Remark: The decision to continue experimentation is made as long as the inequality (31) holds. As soon as (31) is violated, one accepts H_0(i.e. $\theta = \theta_0$) if the function of t in (31) is $< B$, and one rejects H_0 (accepts H_1) if the function of t in (31) is $> A$.

Remark: It should be noted that in the non-replacement case a special problem arises if all n items fail without reaching a decision. This eventuality can be taken care of in various ways and will be discussed in the Appendix.

If we wish to graph the life test data continuously in time, it is convenient to write (31) as

$$(35) \qquad -h_1 + rs < V(t) < h_0 + rs$$

where h_0, h_1, and s are positive constants given by

$$(36) \qquad h_0 = \frac{-\log B}{\frac{1}{\theta_1} - \frac{1}{\theta_0}}, \quad h_1 = \frac{\log A}{\frac{1}{\theta_1} - \frac{1}{\theta_0}}, \text{ and } s = \frac{\log\left(\frac{\theta_0}{\theta_1}\right)}{\frac{1}{\theta_1} - \frac{1}{\theta_0}} .$$

A good way to describe h_0, h_1, and s, is as follows:

h_0 is the intercept on the total life axis of the accept line; $-h_1$ is the intercept on the total life axis of the reject line; and s is the common slope of the two straight lines.

The O.C. curve, i.e., the probability of accepting H_0 when θ is the true parameter value, is given approximately by the parametric equations

$$(37) \qquad L(\theta) = \frac{A^h - 1}{A^h - B^h}, \quad \theta = \frac{\left(\frac{\theta_0}{\theta_1}\right)^h - 1}{h\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)},$$

by letting the parameter h run through all real values. The values of $L(\theta)$ at the five points $\theta = 0, \theta_1, s, \theta_0, \infty$ enables one to sketch the entire curve. These values are, respectively, 0, β, $\log A/(\log A - \log B)$, $1 - \alpha$, and 1. Note that in view of (36), $L(s) = \log A/(\log A - \log B) = h_1/(h_o + h_1)$. An approximate formula for $E_\theta(r)$, the expected number of items to reach a decision, when θ is the mean life is given by

$$
(38) \qquad E_\theta(r) \sim \begin{cases} \dfrac{L(\theta)\log B + [1 - L(\theta)]\log A}{\log(\theta_o/\theta_1) - \theta(\frac{1}{\theta_1} - \frac{1}{\theta_o})} = \dfrac{h_1 - L(\theta)(h_o + h_1)}{s-\theta}, & \theta \neq s \\[2ex] \dfrac{-\log A \log B}{[\log \theta_o/\theta_1]^2} = \dfrac{h_o h_1}{s^2}, & \theta = s. \end{cases}
$$

If we let $k = \theta_o/\theta_1$, the approximate values of $E_\theta(r)$ becomm particularly simple when $\theta = \theta_1$, s, or θ_o.

They are

$$
(39) \qquad \begin{aligned} E_{\theta_1}(r) &\sim [\beta \log B + (1-\beta)\log A] \Big/ [\log k - (k-1)/k] \\ E_s(r) &\sim -\log A \log B/(\log k)^2 \\ E_{\theta_o}(r) &\sim [(1-\alpha)\log B + \alpha \log A] \Big/ [\log k - (k-1)]. \end{aligned}
$$

It can further be shown that $E_\theta(V(t))$ the expected amount of total life observed in reaching a decision is connected with $E_\theta(r)$ by the identity.

$$
(40) \qquad E_\theta(V(t)) = \theta E_\theta(r)
$$

in either the replacement or non-replacement case. Since, in the replacement case $V(t) = nt$, it follows that $E_\theta(t)$, the expected waiting time to reach a decision, is related to $E_\theta(r)$ by the formulae

$$
(41) \qquad E_\theta(t) = \theta E_\theta(r)/n .
$$

In the non-replacement case,

(42) $$E_\theta(t) = \sum_{k=1}^{n} Pr(r=k \mid \theta) E_\theta(X_{k,n})$$

where $E_\theta(X_{k,n})$ is given by (5) . A good approximation for $E_\theta(t)$ is given by

(43) $$E_\theta(t) \sim \theta \log\left(\frac{n}{n-E_\theta(r)}\right) .$$

Practical Applications

It will be convenient to normalize the preceding situation in such a way that $\theta_o = 1$. If this is done it is convenient to calculate once and for all the values of h_o , h_1 , and s for the cases where $\theta_1 = \frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, and for $\alpha = .01$, .05, and $\beta = .01$, .05. In Table 9 , we give the values of h_o , h_1 , and s for each of these cases.

In the event that θ_o is not equal to one and that $k = \theta_o/\theta_1 = \frac{3}{2}$, 2, 3, one can readily find the appropriate equation for $V(t)$ by multiplying h_o , h_1 , and s by θ_o . In Table 10, we give approximate values of $E_\theta(r)$ for the values of α , β , and $k = \theta_o/\theta_1$ given above.

Numerical Examples

1. Find a sequential replacement procedure which will accept a lot with mean life $\theta_o = 1500$ hours, 95% of the time and will reject a lot with mean life $\theta_1 = 500$ hours, 95% of the time. The constant number of items under test is $n = 20$. In this case, $\theta_o = 1500$, $\theta_1 = 500$, $\alpha = \beta = .05$.

Solution: Substituting in formula (31) we get

$$\frac{1}{19} < 3^r e^{-t/37.5} < 19 \quad .$$

In this case (35) becomes

$$-110 + 41r < t < 110 + 41r ,$$

where t represents the length of time that the life test has been in progress and r denotes the number of failures obtained up to time t. The experiment is continued as long as the inequality holds and is stopped as soon as the inequality does not hold. If, at the time of stopping, t is less than the left-hand member of the inequality, we reject $\theta_o = 1500$ (accept $\theta_1 = 500$); if, at the time of stopping, t is greater than the right-hand member of the inequality, we accept $\theta_o = 1500$.

2. Compute $E_\theta(r)$ and $E_\theta(t)$ for $\theta = 0$, $\theta_1 (= 500)$, $s (= 823)$, $\theta_o (= 1500)$, and ∞.

Solution: From Table 10 we get $E_o(r) = 3$, $E_{\theta_1}(r) = 6.14$, $E_s(r) = 7.18$, $E_{\theta_o}(r) = 2.94$, and $E_\infty(r) = 0$.

In the replacement case $E_\theta(t)$ is found most easily for all values of $\theta (\neq \infty)$ by using (41), $E_\theta(t) = \theta E_\theta(r)/n$. This gives $E_o(t) = 0$, $E_{\theta_1}(t) = 155$, $E_s(t) = 295$, and $E_{\theta_o}(t) = 220$. For $\theta = \infty$, the expected waiting time is given by t_∞, where $e^{-t/37.5} = \frac{1}{19}$.

This gives $t_\infty = E_\infty(t) = 110$.

Remark: More generally, in terms of B, n, θ_o, and k we find

$$t_\infty = -\theta_o \log B/n(k-1) .$$

This means that if no items fail by t_∞, we stop experimentation at t_∞ with acceptance of H_o.

3. Assume that we are testing the hypothesis in problem 1. A sample of size

20 is placed on test. Items which fail are replaced at once by new items drawn from the same lot. The experiment is started at time $t = 0$ and the first five failures occur at $x_1 = 20.1$ hours, $x_2 = 100.5$ hours, $x_3 = 121.7$ hours, $x_4 = 167.4$ hours, and $x_5 = 179.2$ hours, all times being measured from $t = 0$.

(a) Verify that no decision has been reached by time x_5.

(b) Verify that if the sixth failure has not yet occurred at 315 hours, measured from $t = 0$, we can stop experimentation at that time with acceptance of H_o, namely that $\theta_o = 1500$.

Solution: We remarked in the solution to (1) that (35) becomes

$$-110 + 41r < t < 110 + 41r \quad .$$

This region is drawn in Figure (2). The life test data are plotted by moving vertically so long as we are waiting for the next failure to occur, and moving horizontally by one unit (in r) at each failure time. In Figure (2) the path crosses into the region of acceptance when $r = 5$, at time $t = 110 + (41)(5) = 315$. Since the sixth failure has not yet occurred we can stop life testing $t = 315$ hours, with acceptance of H_o.

Remark: As a matter of fact, we happen to know in this example that the sixth failure occurs at $x_6 = 346.7$ hours. Thus, as indicated in Figure (2), we saved $346.7 - 315 = 31.7$ hours by virtue of the fact that life test data were becoming available continuously in time.

4. The first six failure times in a sample of 20 (with replacement) are $x_1 = 19.3$, $x_2 = 45.8$, $x_3 = 49.9$, $x_4 = 96.7$, $x_5 = 115.2$, $x_6 = 127.7$. Verify that if the hypotheses being tested are those in Problem 1, then H_o is rejected at time $x_6 = 127.7$ hours.

Solution: x_1, x_2, x_3, x_4, and x_5 all fall within the region bounded by the two straight lines. However, when $r = 6$, $-110 + 41r = 136$. Since $x_6 = 127.7 < 136$, H_o is rejected at time $x_6 = 127.7$ hours. A graphical solution is given in Figure (3) .

Remark: While the acceptance of H_o $(\theta_o = 1500)$ in Problem 3 is made between failure times x_5 and x_6 , rejection of H_o in Problem 4 is made at the failure time x_6 , with an excess over the boundary. This illustrates the point that acceptance of H_o is always made between failure times, whereas rejection of H_o is always made at a failure time.

5. Find a truncated (nonsequential) replacement procedure for testing the hypothesis in Problem 1, using a constant sample size $n = 20$.

Solution: From our earlier results dealing with truncated replacement procedures it can readily be verified that the truncated replacement procedure meeting the requirements is

(i) If $\min\left[x_{10}, 407.5\right] = 407.5$, truncate the experiment at 407.5 with acceptance of H_o .

(ii) If $\min\left[x_{10}, 407.5\right] = x_{10}$, truncate the experiment at x_{10} with acceptance of H_1 .

The O.C. curves of this test procedure and of the one in Problem 1 are essentially the same.

6. Compute $E_\theta(r)$ and $E_\theta(t)$ for the plan in Problem 5 for $\theta = 0, \theta_1, s, \theta_o, \infty$.

Solution: Using the formulae given in the section on truncated replacement procedures and recalling that $E_\theta(t) = \theta E_\theta(r)/n$, one gets $E_\theta(r) = 10$, 9.93 , 8.75 , 5.39 , 0 and

$E_\theta(t) = 0$, 248 , 360 , 404.5 , 407.5 for

$\theta = 0$, $\theta_1(500)$, $s(823)$, $\theta_o(1500)$, and ∞, respectively.

7. Compare $E_\theta(r)$ and $E_\theta(t)$ for the test procedures obtained as solutions to Problems 1 and 5 .

Solution: Using the solutions to Problems 2 and 6 , one has the following comparisons:

$E_\theta(r)$

	$\theta = 0$	$\theta = 500$	$\theta = 823$	$\theta = 1500$	$\theta = \infty$
Truncated with replacement rule	10	9.93	8.75	5.39	0
Sequential rule	3	6.14	7.18	2.94	0

$E_\theta(t)$

	$\theta = 0$	$\theta = 500$	$\theta = 823$	$\theta = 1500$	$\theta = \infty$
Truncated with replacement rule	0	248	360	404.5	407.5
Sequential rule	0	155	295	220	110

These tables give a fairly good idea of the savings associated with adopting a continuous sequential rather than a truncated plan and are typical of what may be expected to happen. A graphical comparison of the two procedures is given in Figure 4 .

8. Find t_∞ in Problem 1 if $\alpha = \beta = .01$.

Solution: $t_\infty = -\theta_0 \log B/n(k-1) = 230$. This is about twice the value of t_∞ when $\alpha = \beta = .05$.

9. Find $L(s)$ for all nine combinations of $\alpha = .01$, .05 , .10 and $\beta = .01, .05$, .10 .

Solution: Since $L(s) = h_1/(h_0 + h_1)$, it follows from Table 9 that $L(s)$ is given by the values in the following Table.

$L(s)$

$\alpha \backslash \beta$	.01	.05	.10
.01	.500	.604	.662
.05	.396	.500	.562
.10	.338	.438	.500

Remark: Since $L(0) = 0$, $L(\theta_0) = 1-\alpha$, $L(\theta_1) = \beta$, and $L(\infty) = 1$ we can readily draw the O.C. curves for all of the cases treated.

10 Find a sequential test for the case when $\alpha = .05$, $\beta = .05$, $\theta_0 = 300$, and $\theta_1 = 100$.

Solution: From Table 9, one finds that $h_0 = h_1 = 1.4722$ (since $\theta_1 = \frac{1}{3}$ if θ_0 is normalized as 1). Therefore the region (35) is given by

$$300(-1.4722 + .5493r) < V(t) < 300(1.4722 + .5493r) .$$

After simplifying this becomes

$$-442 + 165r < V(t) < 442 + 165r .$$

The life test is continued so long as $V(t)$ satisfies both inequalities. As soon as the inequalities are violated, one accepts H_0 (i.e., $\theta_0 = 300$) if $V(t) > 442 + 165r$ and rejects H_0 (accepts H_1 (i.e., $\theta_1 = 100$)) if $V(t) <$ $-442 + 165r$.

Appendix 2 A

Most of the results in sections 1 and 2 of Chapter II are proved in the following reference:

B. Epstein and M. Sobel, "Life Testing", Journal of the American Statistical Association 48, 486-502, 1953.

In Appendix 1 of that paper it is shown that $\hat{\theta}_{r,n}$ as given by (2) is a "best" estimate of θ in the non-replacement case and the p.d.f. (4) is derived. The expected waiting time formula (5) for $E(X_{r,n})$ is derived in Appendix 2 of the reference. The "best" test based on the first r out of n failures having the prescribed properties that its O.C. curve is such that $L(\theta_o) = 1-\alpha$ and $L(\theta_1) \leq \beta$ is obtained directly from the Neyman-Pearson lemma in Appendix 3 of the reference.

There is little point in writing down detailed proofs when they are readily available to the interested reader in the reference just cited. There is, however, good reason to give some supplementary material which is very helpful in understanding the various results. This we shall do in what follows.

In life testing problems where one makes the assumption that the underlying distribution is exponential, the following results play a fundamental role:

(1) Given a Poisson process for which the rate at which events occur per unit time is λ. Let the random function $X(t)$ ($X(0)$ is assumed equal to zero) be the number of events occuring in $(0,t)$. Then

$$\text{(i)} \qquad Pr(X(t) = k) = e^{-\lambda t}(\lambda t)^k/k!, \qquad k = 0,1,2,\ldots .$$

More generally, if $t_2 > t_1$, then

(ii) $$\Pr(X(t_2)-X(t_1) = k) = e^{-\lambda(t_2-t_1)} [\lambda(t_2-t_1)]^k/k!,$$
$$k = 0,1,2,\ldots$$

(iii) Let the random variable T be the waiting time until the first count, or, more generally, the waiting time between successive counts; then the p.d.f. $f(t)$ of T is given by

$$f(t) = \lambda e^{-\lambda t}, \quad t > 0$$
$$= 0, \text{ elsewhere}$$

and the c.d.f. $F(t)$ is given by

$$F(t) = 0, t \leq 0$$
$$= 1-e^{-\lambda t}, \quad t > 0.$$

In Feller's book on Probability Theory the fundamental postulates for the Poisson process are given as follows: whatever the number of changes during $(0,t)$ the (conditional) probability that during $(t,t+h)$ a change occurs is $\lambda h+o(h)$, and the probability that more than one change occurs is of smaller magnitude than h.

<u>Proof:</u> (i) and (ii) are direct consequences of the definitions of a Poisson process. To prove (iii) let T be the random variable representing the waiting time until the first count occurs (measuring time from the origin of time $t = 0$) or the waiting time between two successive counts (where we would now measure time from the moment when the last count was recorded and would wait for the next count), then

(2A.1) $$\Pr(T > t) = \Pr(0 \text{ counts occur in an interval of length } t)$$
$$= \text{Prob}(X(t) = 0) = e^{-\lambda t}.$$

Therefore,

(2A.2) $$F(t) = \Pr(T \leq t) = 1-\Pr(T > t) = 1-e^{-\lambda t}, \quad t > 0$$
$$= 0, \text{ elsewhere}$$

and

(2A.3) $$f(t) = F'(t) = \lambda e^{-\lambda t}, \ t > 0$$
$$= 0, \text{ elsewhere.}$$

(2) A Poisson process has the following interesting feature by definition. If $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq \ldots,$ then the random variables $\{X(t_1), X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})\}$ are mutually independent.

(3) Consider the random variable T distributed with p.d.f. $f(t) = \lambda e^{-\lambda t}, \ t > 0$. Then $\Pr(T > t+\tau | T > t) = e^{-\lambda\tau}$. Put into words: If one observes a Poisson process for a length of time t and no events occur, then the probability of no events occurring in an additional amount of time τ is given by $e^{-\lambda\tau}$. This is a special case of (2) where one considers only the two intervals $(0,t)$ and $(t,t+\tau)$.

(4) Putting (3) into life testing language we have: given that an item has lived for a length of time t, then the conditional probability of surviving an additional τ time units is given by $e^{-\lambda\tau}$. But this is, of course, the probability of an item surviving τ units <u>ab initio</u>. Thus if the underlying distribution is exponential, items that have survived up to any given time are "as good as new" and "have not aged". The proof is very easy:

(2A.4) $$\Pr(T > t+\tau \,|\, T > t) = \Pr(T > t+\tau)/\Pr(T > t).$$
$$= e^{-\lambda(t+\tau)}/e^{-\lambda t} = e^{-\lambda\tau}.$$

This was the result that we wanted to prove.

(5) If n items each having the p.d.f. of life $f(t) = \lambda e^{-\lambda t}$ are placed on test simultaneously at some time $t = 0$, then the first failure to occur, t_1, is exponentially distributed with rate $n\lambda$. Two short proofs follow.

Proof 1: Recalling the connection between Poisson processes and the exponential p.d.f., we can imagine that we superimpose n Poisson processes each having failure rate λ. The result is a Poisson process with failure rate $n\lambda$. Hence from (1)(iii) the p.d.f. of the waiting time for the first failure is given by $f_1(t) = n\lambda e^{-n\lambda t}$, $t > 0$ and the c.d.f. is $F_1(t) = 1-e^{-n\lambda t}$, $t > 0$.

Proof 2: An alternative proof is to recall that the first failure is the smallest in a sample of size n drawn from an exponential p.d.f. The c.d.f. of the smallest value is given by
$F_1(t) = 1-(1-F(t))^n = 1-e^{-n\lambda t}$, $t > 0$. This is the result obtained before.

We now use these results to obtain formulae (2) through (6) in Section 1. First we note that given a Poisson process with rate λ, then the associated waiting time random variable T has expectation

$$E(T) = \int_0^\infty \lambda t e^{-\lambda t} dt = \frac{1}{\lambda}. \tag{2A.5}$$

In the life testing context, where T is thought of as life, $E(T) =$ mean life $= \Theta = \frac{1}{\lambda}$. Thus items exponentially distributed with mean life Θ can be thought of as waiting times between successive occurrences of a Poisson process with rate $\lambda = \frac{1}{\Theta}$.

We now obtain formula (2) (non-replacement case) using the ideas of Poisson processes. Placing n items on test at time $t = 0$, with each item having an exponential p.d.f. with mean life Θ, is equivalent to considering the superposition of n Poisson processes, each having rate $\lambda = \frac{1}{\Theta}$. The process obtained by superposition is still Poisson with parameter $\lambda_n = \frac{n}{\Theta}$. The first failure observed at time $x_{1,n}$ is exponentially distributed with mean life $\frac{\Theta}{n}$ and so $nx_{1,n}$ is exponentially distributed with mean life Θ. Consider now what happens after time $x_{1,n}$. At time $x_{1,n}$ one has $(n-1)$ items left each with mean life Θ (this is a consequence of 3 and 4). Thus one is now dealing with a superposition of $(n-1)$ Poisson processes each having rate $\frac{1}{\Theta}$, and hence the superposition is a Poisson process with rate $(\frac{n-1}{\Theta})$. Therefore, $x_{2,n}-x_{1,n}$ is exponentially distributed with mean life $\frac{\Theta}{n-1}$ and $(n-1)(x_{2,n}-x_{1,n})$ is exponentially distributed with mean life Θ. Also, $x_{1,n}$ and $(x_{2,n}-x_{1,n})$ are mutually independent. Continuing in the same way $\left\{x_{1,n};\ x_{2,n}-x_{1,n};\ldots;\ x_{i,n}-x_{i-1,n};\ldots,\ x_{r,n}-x_{r-1,n}\right\}$ are mutually independent and drawn from exponential distributions having mean lives $\left\{\frac{\Theta}{n}, \frac{\Theta}{n-1}, \ldots, \frac{\Theta}{n-i}, \ldots, \frac{\Theta}{n-r+1}\right\}$, respectively. More simply the r random variables $\left\{y_i = (n-i+1)(x_i-x_{i-1}),\ i = 1,2,\ldots,r\right\}$ where $x_0 = 1$ are mutually independent with common p.d.f. $\frac{1}{\Theta}e^{-y/\Theta}$, $y > 0$. Therefore,

$$\hat{\Theta} = \sum_{i=1}^{r} y_i/r = \sum_{i=1}^{r} (n-i+1)(x_i-x_{i-1})/r \tag{2A.6}$$

$$= \left[\sum_{i=1}^{r} x_{i,n} + (n-r)x_{r,n}\right] \Big/ r$$

is unbiased. The other properties (such as maximum likehood, unbiasedness, minimum variance, efficiency, and sufficiency) are proved in the reference cited above. The p.d.f. (4) of $\hat{\Theta}$ follows directly from the fact that the sum of independent random variables each of which is exponential follows a Type III distribution. More precisely consider the random variable $U = \sum_{i=1}^{r} y_i$. This can be considered as the waiting time for the r^{th} event in a Poisson process with parameter $\lambda = \frac{1}{\Theta}$. The p.d.f.

of U is found by using the fundamental postulates for a Poisson process. Thus

(2A.7) $\Pr(t < U < t+\Delta t) = \Pr(r-1$ events occur in $(0,t)$ and 1 event occurs in $t, t+\Delta t)$

$= \Pr[r-1$ events occur in $(0,t)]$. $\Pr[1$ event occurs in $(t, t+\Delta t)]$

$$= \frac{(\frac{t}{\theta})^{r-1} e^{-\frac{t}{\theta}}}{(r-1)!} \frac{\Delta t}{\theta} .$$

Therefore, the p.d.f. of U is given by

(2A.8) $$h(t) = \frac{1}{(r-1)!} \frac{1}{\theta} (\frac{t}{\theta})^{r-1} e^{-t/\theta}, \quad t > 0.$$

But $\hat{\theta}_{r,n} = U/r$ and using simple transformations the p.d.f. of $\hat{\theta}_{r,n}$ becomes (4). To prove (5), we note that

(2A.9) $$X_{r,n} = X_{1,n} + (X_{2,n}-X_{1,n}) + \ldots + (X_{r,n}-X_{r-1,n})$$

$$= \frac{Y_1}{n} + \frac{Y_2}{n-1} + \ldots + \frac{Y_r}{n-r+1} .$$

But the Y_i are each distributed with the p.d.f. $\frac{1}{\theta} e^{-x/\theta}$. Therefore,

(2A.10) $$E(X_{r,n}) = \theta \left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{n-r+1}\right) = \theta \sum_{j=1}^{r} 1/n-j+1$$

and thus (5) holds.

Incidentally, since Y_i's are also mutually independent it follows that

(2A.11) $$\operatorname{Var}(X_{r,n}) = \theta^2 \sum_{j=1}^{r} 1/(n-j+1)^2 ;$$

also

(2A.12) $$\operatorname{cov}(X_{r,n}; X_{s,n}) = \operatorname{var} X_{r,n} = \theta^2 \sum_{j=1}^{r} 1/(n-j+1)^2, \quad \text{if } s \geq r$$

$$= \operatorname{var} X_{s,n} = \theta^2 \sum_{j=1}^{s} 1/(n-j+1)^2, \quad \text{if } r \geq s.$$

For example, suppose $s \geq r$. Then

$$X_{s,n} = X_{r,n} + (X_{s,n} - X_{r,n}). \tag{2A.13}$$

Hence

$$\begin{aligned} \operatorname{cov}(X_{r,n}; X_{s,n}) &= \operatorname{cov}(X_{r,n}; X_{r,n} + (X_{s,n} - X_{r,n})) \\ &= \operatorname{cov}(X_{r,n}; X_{r,n}) + \operatorname{cov}(X_{r,n}; X_{s,n} - X_{r,n}). \end{aligned} \tag{2A.14}$$

Noting that $X_{r,n}$ (the waiting time for the r^{th} failure) and $X_{s,n} - X_{r,n}$ (the waiting time between the r^{th} and s^{th} failure) are independent, we get that

$$\operatorname{cov}(X_{r,n}; X_{s,n}) = \operatorname{Var}(X_{r,n}) \quad \text{if} \quad s \geq r .$$

Similarly

$$\operatorname{cov}(X_{r,n}; X_{s,n}) = \operatorname{Var}(X_{s,n}) \quad \text{if} \quad r \geq s.$$

Thus (2A.12) is proved.

Up to this point we have dealt exclusively with the non-replacement case. If items are replaced as they fail, then it is clear that placing n items on test and replacing failed items at once by new items is equivalent to observing a Poisson process with rate $\lambda = n/\Theta$. If $x_{1,n} \leq x_{2,n} \leq \cdots \leq x_{r,n}$ are the first r failure times (time being measured from the beginning of the experiment), then $\left\{x_{1,n}; x_{2,n} - x_{1,n}, \ldots, x_{r,n} - x_{r-1,n}\right\}$ are mutually independent and idencally distributed with common p.d.f. $\frac{n}{\Theta} e^{-nx/\Theta}$. The random variables $\left\{z_i = n(x_i - x_{i-1}),\ i = 1,2,\ldots,r\right\}$, where $x_0 = 0$, are mutually independent with common p.d.f. $\frac{1}{\Theta} e^{-z/\Theta}$, $z > 0$. Therefore,

$$\hat{\Theta} = \sum_{i=1}^{r} z_i/r = n x_{r,n}/r \tag{2A.15}$$

is unbiased. Other optimum properties are easy to show and as before the p.d.f. of $\hat{\Theta}$ is given by (4).

To prove (6) note that

(2A.16) $$X_{r,n} = X_{1,n} + (X_{2,n} - X_{1,n}) + \ldots + (X_{r,n} - X_{r-1,n})$$

$$= \frac{1}{n} \sum_{i=1}^{r} Z_i .$$

But the Z_i's are each distributed with p.d.f. $\frac{1}{\theta} e^{-x/\theta}$.

Therefore,

(2A.17) $$E(X_{r,n}) = \frac{r}{n} E(Z) = \frac{r\theta}{n}.$$

From the mutual independence of the Z_i's it also follows that

(2A.18) $$\text{Var } X_{r,n} = \frac{r\theta^2}{n^2}$$

and

(2A.19) $$\text{cov } (X_{r,n}; X_{s,n}) = \text{var } X_{r,n} = \frac{r\theta^2}{n^2}, \quad \text{if } s \geq r$$

$$= \text{var } X_{s,n} = \frac{s\theta^2}{n^2}, \quad \text{if } r \geq s.$$

Appendix 2B

We have seen that in life tests where items that fail are not replaced, then the statistic $\hat{\theta}_{r,n}$, where

$$(2B.1) \qquad \hat{\theta}_{r,n} = \frac{x_{1,n}+x_{2,n}+\ldots+x_{r-1,n}+(n-r+1)x_{r,n}}{r},$$

is a "best" estimate. It can further be shown that the "best" test for $\theta = \theta_0$ against alternatives $\theta < \theta_0$ is given by an acceptance region $\hat{\theta}_{r,n} > C$. If the Type I error is controlled at α, then $C = \theta_0 \chi^2_{1-\alpha}(2r)/2r$. Inspection of this statistic reveals that if $n > r$, $x_{r,n}$ is weighted more heavily than the earlier observations $x_{1,n}, x_{2,n}, \ldots, x_{r-1,n}$. This would lead one to suspect that estimates based only on $x_{r,n}$ (i.e., the r^{th} failure time only) may be highly efficient when compared with estimates based on $\hat{\theta}_{r,n}$ and further that rules of action based on $x_{r,n}$ have O.C. curves very close to those based on $\hat{\theta}_{r,n}$. This question has been studied in detail in the estimation case in a report by B. Epstein entitled "Estimates of mean life based on the r^{th} smallest value in a sample of size n drawn from an exponential distribution", Wayne University Technical Report No. 2, July, 1952. It is shown in this report that a highly efficient estimate of θ is given by $\beta_{r,n}x_{r,n}$ where $\beta_{r,n} = 1 \Big/ \sum_{j=1}^{r} 1/n-j+1$. $\beta_{r,n}x_{r,n}$ is an unbiased estimate of θ. It can be verified readily that

$$(2B.2) \qquad \mathrm{Var}\,(\beta_{r,n}\, x_{r,n}) = K_{r,n}\theta^2$$

where

$$(2B.3) \qquad K_{r,n} = \sum_{j=1}^{r} 1/(n-j+1)^2 \Big/ \Big(\sum_{j=1}^{r} 1/n-j+1\Big)^2 .$$

Efficiency of the estimator $\beta_{r,n}x_{r,n}$ relative to $\hat{\theta}_{r,n}$ is given by

$$E_{r,n} = \text{Var}\,\hat{\theta}_{r,n} / \text{var}\,(\beta_{r,n} x_{r,n}) = \frac{1}{rK_{r,n}} \tag{2B.4}$$

In the report to which we just referred, tables are given for $\beta_{r,n}, K_{r,n}$ and $E_{r,n}$ for $n = 1(1)20(5)30(10)100$ and $r = 1(1)n$. An inspection of these tables reveals that $E_{r,n} \geq .9$ for $\frac{r}{n} < \frac{2}{3}$. Furthermore it is shown that

$E_{r,n} \geq .9990$	if	$\frac{r}{n} = .1$
" $\geq .9960$	if	" $= .2$
" $\geq .9893$	if	" $= .3$
" $\geq .9784$	if	" $= .4$
" $\geq .9608$	if	" $= .5$
" $\geq .9329$	if	" $= .6$
" $\geq .8874$	if	" $= .7$
" $\geq .8094$	if	" $= .8$
" $\geq .6548$	if	" $= .9$

It has similarly been observed that the O.C. curve resulting from using the acceptance region $\beta_{r,n} x_{r,n} > C$ virtually coincides with the O.C. curve associated with $\hat{\theta}_{r,n} > C$.

Bibliography for Sections 1 and 2 of Chapter II

1. B. Epstein and M. Sobel, "Some tests based on the first r ordered observations drawn from an exponential distribution", Stanford University Technical Report No. 6, Wayne University Technical Report No. 1, March, 1952.
2. B. Epstein, "Estimates of mean life based on the r^{th} smallest value in a sample of size n drawn from an exponential distribution", Wayne University Technical Report No. 2, July, 1952.
3. B. Epstein and M. Sobel, "Life Testing", Journal of the American Statistical Association 48, 486-502, 1953.
4. W. Feller, "An Introduction to Probability Theory and its Applications", Vol. I, Second Ed; John Wiley and Sons, 1956. (See particularly Chapter 17.)
5. "Tables of the Cumulative Binomial Distribution", Vol. 35, Annals of the Computation Laboratory of Harvard University, 1955.

Appendix 2 C

A detailed discussion of truncated replacement and non-replacement tests is given in the following paper:

B. Epstein, "Truncated Life Tests in the Exponential Case", Annals of Mathematical Statistics **25**, 555-564, 1954.

Section 2 of this paper, pp. 555-558, gives proofs of formulae (21) through (28) inclusive.

Appendix 2 D

Tests of the form $\hat{\theta}_{r,n} > C$ considered as truncated tests.

The following material follows very closely section 3 of the paper cited in Appendix 2 C.

We have seen that when testing $H_o : \theta = \theta_o$ against any simple alternative $\theta = \theta_1 (\theta_1 < \theta_o)$, the "best" region of acceptance for H_o (in the sense of Neyman and Pearson), based on the first r out of n ordered observations from an exponential distribution, is of the form

$$\hat{\theta}_{r,n} > C, \text{ where } \hat{\theta}_{r,n} = [\sum_{i=1}^{r} x_{i,n} + (n-r)x_{r,n}]/r \text{ in the}$$

non-replacement case and $\hat{\theta}_{r,n} = nx_{r,n}/r$ in the replacement case.

One could interpret the decision rule $\hat{\theta}_{r,n} > C$ to mean that we wait until time $x_{r,n}$ (the time when the r^{th} failure occurs), then compute $\hat{\theta}_{r,n}$ and make the appropriate decision. However, in the event that we are able to observe the life test continuously, this clearly wastes information. Indeed, we assert that, if continuous observation is taken into account, we can frequently shorten the waiting time to reach a decision and reduce the number of items failed. To see this we note that $\hat{\theta}_{r,n} > C$ becomes $[\sum_{i=1}^{r} x_{i,n} + (n-r)\, x_{r,n}] > rC$ in the non-replacement case and $nx_{r,n} > rC$ in the replacement case.

But $[\sum_{i=1}^{r} x_{i,n} + (n-r)\, x_{r,n}]$ is the total observed life up to time $x_{r,n}$ in the non-replacement case (note that $\sum_{i=1}^{r} x_{i,n}$ is the total life of the r items which failed and $(n-r)x_{r,n}$ is the amount of time lived by the $(n-r)$ items which did not fail) and $nx_{r,n}$ is the total observed life up to time $x_{r,n}$ in the replacement case (note that in the replacement case n items are constantly on test for a length of time $x_{r,n}$). Thus accepting H_o when $\hat{\theta}_{r,n} > C$ is equivalent to accepting H_o if the total life observed up to time $x_{r,n}$ is greater than rC. Suppose now that at some moment t there are exactly k failures, $0 \le k \le r-1$, and that the observed total life $V(t)$ given by $V(t) = \sum_{i=1}^{k} x_{i,n} + (n-k)t$ in the non-replacement case and by $V(t) = nt$ in the replacement case exceeds rC (Note that in the non-replacement case $\sum_{i=1}^{k} x_{i,n}$ is the amount contributed to $V(t)$ by the k items which failed by time t and $(n-k)t$ is the amount contributed by the $(n-k)$ items which have not failed. In particular, if $t = x_{r,n}$, then $V(x_{r,n}) = \sum_{i=1}^{r} x_{i,n} + (n-r)x_{r,n}$. The formula for $V(t)$ in the replacement case is obvious.) Since $V(t)$ is monotonically increasing in t, we know that $V(x_{r,n}) \ge V(t) > rC$, and therefore we should stop the life test at time t and accept H_o. More generally a decision rule having precisely the same O.C. curve as $\hat{\theta}_{r,n} > C$ but requiring on the average fewer failures and a shorter decision time is based on terminating at total observed life $= \min (V(x_{r,n}), rC)$ (where both r and C are preassigned). If the experiment is terminated at total life $V(x_{r,n})$ (i.e., if the total life required to observe r failures is $< rC$), then the action in terms of hypothesis testing is the rejection of the null hypothesis. If life testing is terminated with total life $= rC$ (i.e., if $V(x_{r,n})$, the total life required to observe r failures, exceeds rC), then the action taken is to accept the null hypothesis. (Note that in the replacement case $(V(x_{r,n}), rC)$ becomes $\min (x_{r,n}; \frac{rC}{n})$ where $x_{r,n}$ is the time of the r^{th} failure and $\frac{rC}{n}$ is a truncation time.)

Described in more detail the decision rule is as follows:

(a) Continue life testing so long as $V(t) < rC$ and $0 \leq k \leq r-1$.

(b) Stop experimentation at time t with acceptance of H_o as soon as $V(t) > rC$ and $0 \leq k \leq r-1$.

(c) Stop experimentation at time $x_{r,n}$ with rejection of H_o if $V(t) < rC$ for all $t \leq x_{r,n}$. (Note that acceptance of H_o takes place between failure times, and always before time $x_{r,n}$.)

We now proceed to find some useful properties of the truncated rule based on $V(t)$. To find these properties, we remark that (defining $x_{o,n}$ as zero)

$$(2D.1) \qquad \sum_{i=1}^{r} x_{i,n} + (n-r)\, x_{r,n} = \sum_{i=1}^{r} (n-i+1)(x_{i,n}-x_{i-1,n})$$

in the non-replacement case and

$$(2D.2) \qquad nx_{r,n} = \sum_{i=1}^{r} n(x_{i,n}-x_{i-1,n})$$

in the replacement case.

Introducing (as was done in Appendix 2A) new random variables defined by

$$(2D.3) \qquad y_1 = nx_{1,n} \text{ and } y_i = (n-i+1)(x_{i,n}-x_{i-1,n}),\ i = 2,3,\ldots,r$$

in the non-replacement case and

(2D.4) $$y_1 = nx_{1,n} \quad \text{and} \quad y_i = n(x_{i,n} - x_{i-1,n}),\; i = 2,3,\ldots,r$$

in the replacement case.

$V(t) > rC$ can be written as

(2D.5) $$\sum_{i=1}^{r} y_i > rC.$$

We saw in Appendix 2A that the y_i are mutually independent random variables, each distributed with common p.d.f. $\frac{1}{\theta} e^{-x/\theta}$, $x > 0$, $\theta > 0$. If we interpret y_i as the time interval between the $(i-1)^{st}$ and i^{th} event in a Poisson process having mean occurrence rate $\lambda = 1/\theta$, it is clear that $\sum_{i=1}^{r} y_i > rC$, if and only if k, the number of events in a time interval of length rC, is $0 \leq k \leq r-1$. If the number of events in such an interval is $\geq r$, then $\sum_{i=1}^{r} y_i \leq rC$. Thus the probability of reaching a decision requiring exactly $\rho = k$ failures is

(2D.6) $$\Pr(\rho = k \mid \theta) = p(k;\mu_\theta),\; k = 0,1,2,\ldots,r-1$$

$$\Pr(\rho = r \mid \theta) = 1 - \sum_{k=0}^{r-1} p(k;\mu_\theta) = 1 - \Pi(r-1;\mu_\theta),$$

where $\mu_\theta = rC/\theta$. The expected number of observations to reach a decision is given by

(2D.7) $$E_\theta(\rho) = \sum_{k=0}^{r} k\Pr(\rho = k \mid \theta) = \mu_\theta \Pi(r-2;\mu_\theta) + r[1-\Pi(r-1;\mu_\theta)].$$

$E_\theta(V(t))$, the expected total life in reaching a decision, is given by

(2D.8) $$E_\theta(V(t)) = \theta E_\theta(\rho).$$

$E_\theta(T)$, the expected waiting time to reach a decision, is given by

(2D.9) $$E_\theta(T) = \sum_{k=1}^{r} \Pr(\rho = k|\theta)\, E_\theta(X_{k,n})$$

where $E_\theta(X_{k,n}) = \theta \sum_{j=1}^{k} \frac{1}{n-j+1}$ in the non-replacement case and $E_\theta(X_{k,n}) = k\theta/n$ in the replacement case. In the replacement case

(2D.10) $$E_\theta(T) = \sum_{k=1}^{n} \Pr(\rho = k|\theta) k\frac{\theta}{n}$$
$$= \frac{\theta}{n} E_\theta(\rho) = \frac{E_\theta(V(t))}{n} .$$

Finally $L(\theta)$, the probability of accepting $\theta = \theta_0$ when θ is true, is given by $L(\theta) = \pi(r-1;\mu_\theta)$. Note that in the replacement case (25) through (28) coincide with what we have just done if we set $T_0 = rC/n$ and $r = r_0$. If this is done then $\lambda_\theta = nT_0/\theta = rC/\theta = \mu_\theta$.

Remark: In the above we considered a test based on $\hat{\theta}_{r,n}$ as a truncated test. This involved consideration of total life. The essence of what was said is a special case of the following: Suppose that the experimenter wishes to expend no more than total life V^* in experimentation and that he employs the following rule of action: Reject if r_0 failures occur before total life V^* has been used up; accept if fewer than r_0 failures occur by the time one has observed a total life of V^*. In the event that one rejects, experimentation stops at $V(\tau_{r_0})$, the total life observed up to and including τ_{r_0}, the r_0th failure time. In the event that one accepts, the total life observed will be V^*. It follows directly from the properties of Poisson processes that the probability of reaching a decision requiring exactly $\rho = k$ failures is

(2D.11) $\Pr(\rho = k|\theta) = p(k;\mu_\theta)$, $k = 0,1,2,\ldots,r_0-1$

and

$$\Pr(\rho = r_0 | \theta) = 1 - \sum_{k=0}^{r_0-1} p(k;\mu_\theta) = 1 - \pi(r_0-1,\mu_\theta),$$

where $\mu_0 = V^*/\theta$. The expected number of observations to reach a decision is given by

(2D.12) $$E_\theta(\rho) = \mu_\theta \pi(r-2;\mu_\theta) + r\,[1 - \pi(r-1;\mu_\theta)].$$

$E_\theta(V(t))$, the expected total life in reaching a decision is given by

(2D.13) $$E_\theta(V(t)) = \theta E_\theta(\rho)$$

and $L(\theta)$, the probability of accepting $\theta = \theta_0$ when θ is true, is given by

(2D.14) $$L(\theta) = \sum_{k=0}^{r_0-1} p(k;\mu_\theta) = \pi(r_0-1;\mu_\theta).$$

The considerations involving $\hat{\theta}_{r,n}$ are a special case of what we have just done, with $V^* = rC$ and $r_0 = r$.

Appendix 2 E

As an illustration of the theory presented in Section 3 and Appendix 2 D we consider three test procedures which have virtually the same operating characteristic curve. Specifically it is assumed that we wish to test $H_0:\theta_0 = 1500$ hours against $H_1:\theta_1 = 500$ hours with $\alpha = \beta = .05$; i.e., we want $L(\theta_0) = 1-\alpha = .95$ and $L(\theta_1) = \beta = .05$ (actually we have to be satisfied with $L(\theta_1) \leq .05$). The three procedures are:

(a) 20 items are taken at random from the lot and placed on life test. Items which fail are not replaced. At each moment t, compute the total life

$$V(t) = \sum_{i=1}^{k} x_{i,n} + (n-k)\,t,$$ where k is the number of failures which have occurred before time t and $n = 20$ (if $i = 0$, define total life as nt). If $V(t)$ exceeds 8150 for any k, $0 \leq k \leq 9$, stop the experiment at time t and accept H_0 ($\theta = 1500$ hours). Otherwise

the action taken is to reject. This test is equivalent to accepting H_o if $\hat{\theta}_{10,20} > 815$ and rejecting H_o if $\hat{\theta}_{10,20} < 815$. (From Table 5 we see that if $\theta_o/\theta_1 = 3, \alpha = \beta = .05$, then $r = 10$ and $\chi^2_{1-\alpha}(2r)/2r = .5426$. Therefore the acceptance region is $\hat{\theta}_{10,20} > (1500(.5426) = 815.)$

(b) 20 items are taken at random from the lot and placed on test. Failed items are not replaced. If min $[X_{10,20}, 540] = 540$ (i.e., the tenth failure occurs <u>after</u> 540 hours), truncate the experiment at 540 hours with acceptance of H_o. If min $[X_{10,20}, 540] = X_{10,20}$ (i.e., the tenth failure occurs <u>before</u> 540 hours), truncate the experiment at $X_{10,20}$ with the rejection of H_o. (From Table 4a, using $r = 10$, $n = 2r = 20$, we see that the truncation time $T_o = 1500(.363) = 540$.)

(c) 20 items are taken at random from the lot and placed on test. An item which fails is replaced at once by a new item from the original lot. The time $X_{i,n}$ when the i^{th} failure occurs is measured from the beginning of experimentation. If min $[X_{10,20}, 407.5] = 407.5$, truncate the experiment at 407.5 hours with the acceptance of H_o. If min $[X_{10,20}, 407.5] = X_{10,20}$, truncate the experiment at $X_{10,20}$ with the rejection of H_o. (In the replacement case the truncation time T_o is given by $\theta_o \chi^2_{1-\alpha}(2r)/2n$. This gives 407.5 for the value in this problem.)

In the table below we give $L(\theta)$, $E_\theta(r)$, and $E_\theta(T)$ for the tests A, B, and C for selected values of θ.

Properties of Three Test Procedures

Mean Life θ	$L(\theta)$			$E_\theta(r)$			$E_\theta(T)$		
	A	B	C	A	B	C	A	B	C
250	.0000	.0000	.0000	10	10	10	167.2	167.2	125.0
500	.038	.043	.038	9.93	9.94	9.93	331.4	331.6	248.3
750	.355	.366	.355	9.10	9.25	9.10	444.7	453.5	341.3
1000	.698	.702	.698	7.68	8.06	7.68	481.8	509.1	384.0
1250	.876	.877	.876	6.39	6.93	6.39	484.8	529.2	399.3
1500	.950	.950	.950	5.39	6.02	5.39	474.7	536.0	404.5
1750	.979	.979	.979	4.64	5.30	4.64	466.0	538.3	406.3
2000	.991	.991	.991	4.07	4.73	4.07	458.3	539.4	407.0
2250	.996	.995	.996	3.62	4.27	3.62	452.3	539.7	407.3
2500	.998	.993	.998	3.26	3.88	3.26	447.3	539.9	407.4

Bibliography for Section 3 of Chapter II.

1. B. Epstein, "Truncated Life Tests in the Exponential Case," Annals of Mathematical Statistics, 25, 555-564, 1954.

2. B. Epstein, "Statistical Problems in Life Testing," Proceedings of the Seventh Annual Convention of the American Society for Quality Control, 385-398, 1953.

3. E. C. Molina, Poisson's Experimental Binomial Limit, D. Van Nostrand, 1949.

4. "Tables of the Cumulative Binomial Distribution, Vol. 35, Annals of the Computation Laboratory of Harvard University, 1955.

Appendix 2F

Detailed proofs of the results on sequential life tests sketched in Section 4 of Chapter II are given in the following reference: B. Epstein and M. Sobel, "Sequential Life Tests in the Exponential Case," Annals of Mathematical Statistics 26, 82-93, 1955. In Section 2 of this reference one will find a derivation of formulae (31) through (39) inclusive. In Section 3 of the reference the basic identity $E_\theta(V(t)) = \theta\, E_\theta(r)$, relating the expected moment of total life observed in reaching a decision and the expected number of failures, is derived. This formula holds in general, whether or not items on test are replaced.

Appendix 2G

1. Introduction

It is interesting to ask the question: How will truncation of the sequential life test affect the Type I error α and Type II error β ? We know, from considerations analogous to those of A. Wald, that the sequential life test procedure based on using (31) will eventually terminate. But this may be inordinately expensive in terms of either the time involved in the life test, or in terms of the number of items failed, or both. There are many situations where it is desirable and even necessary that we place a definite upper limit on either the number of items failed or on the total length of the life test (or, if necessary on both). In what follows we study how much one changes the Type I and Type II errors, if one truncates the sequential life test in ~~[illegible]~~ various ways.

Remark: From this point on we follow closely considerations in Wald's book, pp. 61-65.

2. Truncation on the number of items failed.

Suppose first that we set a definite limit, r_o, on the number of items failed. We can achieve this by truncating the sequential life test at $r = r_o$, i.e., by giving a new rule for the acceptance or rejection of $H_o: \theta = \theta_o$ when r_o failures have occurred if the sequential life test did not lead to a decision for $r \leq r_o$. A simple and reasonable truncation rule after the r_o'th failure is the following: If the sequential probability ratio test given by (31) does not lead to a decision for $r \leq r_o$, accept $H_o: \theta = \theta_o$ after the r_o'th failure has occurred if

$$\text{(2G.1)} \qquad \log B < r_o \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(x_{r_o}) \leq 0$$

and reject H_o (accept $H_1: \theta = \theta_1$) after the r_o'th failure has occurred if

$$\text{(2G.2)} \qquad 0 < r_o \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(x_{r_o}) < \log A \quad .$$

Truncating the sequential life test after r_o items fail will change the Type I and Type II errors. They will no longer be α and β, the Type I and Type II errors, respectively, in the untruncated sequential case. The effect of the truncation on the α and β depends, of course, on r_o.

The larger one makes r_o, the smaller are the truncation effects on α and β. Let us denote the resulting Type I and Type II errors as $\alpha(r_o)$ and $\beta(r_o)$, respectively, if the sequential life test is truncated at $r = r_o$ failures at the latest. We now derive upper bounds for $\alpha(r_o)$ and $\beta(r_o)$.

To obtain an upper bound for $\alpha(r_o)$ we have to consider the cases in which the truncated life test leads to the rejection of $H_o: \theta = \theta_o$, while the non-truncated sequential life test leads to the acceptance of H_o.

Supposing that H_o holds (i.e., that $\theta = \theta_o$), let $\rho_o(r_o)$ be the probability that the sample random function associated with a life test is such that the truncated life test leads to rejection of H_o, while the non-truncated life test leads to the acceptance of H_o. Clearly we see that

$$\alpha(r_o) \leq \alpha + \rho_o(r_o) \quad . \tag{2G.3}$$

The reason for the inequality rather than the equality is that there may be sample random functions associated with life tests for which the truncated life test leads to acceptance of H_o, while the non-truncated life test leads to the rejection of H_o. To obtain an upper bound for $\alpha(r_o)$, we need merely derive an upper bound for $\rho_o(r_o)$. Assuming that H_o is true, $\rho_o(r_o)$ is the probability that the random function associated with a life test is such that the following three conditions hold simultaneously:

(2G.4)

(i) $\log B < r \log \frac{\theta_o}{\theta_1} - (\frac{1}{\theta_1} - \frac{1}{\theta_o}) V(t) < \log A$, for $r = 1,2,\ldots,r_o-1$

and for all $t < x_{r_o}$;

(ii) $0 < r_o \log \frac{\theta_o}{\theta_1} - (\frac{1}{\theta_1} - \frac{1}{\theta_o}) V(x_{r_o}) < \log A$;

and

(iii) When the sequential life test is continued beyong the r_o'th failure, it terminates with the acceptance of H_o.

Assuming that H_o is true, let $\bar{\rho}_o(r_o)$ be the probability that condition (ii) holds, i.e.,

$$\bar{\rho}_o(r_o) = Pr \left\{ 0 < r_o \log \frac{\theta_o}{\theta_1} - (\frac{1}{\theta_1} - \frac{1}{\theta_o}) V(x_{r_o}) < \log A \,|\, \theta = \theta_o \right\} . \tag{2G.5}$$

Since the probability that (ii) is fulfilled cannot be smaller than the probability that conditions (i) , (ii) and (iii) are fulfilled simultaneously, we have

(2G.6) $$\bar{\rho}_o(r_o) \geq \rho_o(r_o)$$

and therefore,

(2G.7) $$\alpha(r_o) \leq \alpha + \bar{\rho}_o(r_o)$$

Thus $\alpha + \bar{\rho}_o(r_o)$ is an upper bound for $\alpha(r_o)$. We show further on that $\bar{\rho}_o(r_o)$ can be computed easily.

To obtain an upper bound for $\beta(r_o)$, let us assume that $H_1: \theta = \theta_1$ is true and let $\rho_1(r_o)$ then be the probability that the truncated life test leads to the acceptance of H_o while the non-truncated life test leads to the rejection of H_o . In other words, $\rho_1(r_o)$ is the probability (assuming that $\theta = \theta_1$ is true) that the sample random function associated with a life test is such that the following three conditions hold simultaneously:

(i) $\log B < r \log \frac{\theta_o}{\theta_1} - (\frac{1}{\theta_1} - \frac{1}{\theta_o}) V(t) < \log A$, for $r = 1,2,\ldots,r_o - 1$

and for all $t < x_{r_o}$;

(2G.8)

(ii) $\log B < r_o \log \frac{\theta_o}{\theta_1} - (\frac{1}{\theta_1} - \frac{1}{\theta_o}) V(x_{r_o}) \leq 0$;

(iii) If the sequential life test is continued beyond the r_o'th failure, it terminates with the acceptance of $H_1: \theta = \theta_1$.

Clearly

(2G.9) $$\beta(r_o) \leq \beta + \rho_1(r_o) \quad .$$

Since it is difficult to determine $\rho_1(r_o)$, we give a simple upper bound first. Assuming that H_1 is true, let $\bar{\rho}_1(r_o)$ be the probability that condition (ii) holds, i.e.,

$$(26.10) \qquad \bar{\rho}_1(r_o) = \Pr\left\{\log B < r_o \log\frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(x_{r_o}) \leq 0 \;\middle|\; \theta = \theta_1\right\} .$$

Then $\bar{\rho}_1(r_o) \geq \rho_1(r_o)$ and hence

$$(26.11) \qquad \beta(r_o) \leq \beta + \bar{\rho}_1(r_o) \quad .$$

We now show how to compute $\bar{\rho}_o(r_o)$ and $\bar{\rho}_1(r_o)$. To compute $\bar{\rho}_o(r_o)$, we recall that if r_o is preassigned, then, under the hypothesis that $H_o: \theta = \theta_o$ is true, $2V(X_{r_o})/\theta_o$ is distributed as $\chi^2(2r_o)$. Consequently,

$$(26.12) \qquad \bar{\rho}_o(r_o) = \Pr\left\{\frac{2(r_o \log k - \log A)}{k-1} < \chi^2(2r_o) < \frac{2r_o \log k}{k-1}\right\}$$

where $k = \theta_o/\theta_1$.

In a similar way one can compute $\bar{\rho}_1(r_o)$. If r_o is preassigned, then, under the hypothesis that $H_1: \theta = \theta_1$ is true, $2V(x_{r_o})/\theta_1$ is distributed as $\chi^2(2r_o)$. Consequently

$$(26.13) \qquad \bar{\rho}_1(r_o) = \Pr\left\{\frac{2r_o k \log k}{k-1} \leq \chi^2(2r_o) < \frac{2k(r_o \log k - \log B)}{k-1}\right\}$$

Thus we can summarize our results as follows:

$$(26.14) \qquad \alpha(r_o) \leq \alpha + \Pr\left\{\frac{2(r_o \log k - \log A)}{k-1} < \chi^2(2r_o) < \frac{2r_o \log k}{k-1}\right\}$$

and [redacted]

$$\beta(r_o) < \beta + \Pr\left\{\frac{2r_o\, k \log k}{k-1} \leq \chi^2(2r_o) < \frac{2k(r_o \log k - \log B)}{k-1}\right\} . \tag{20.15}$$

It is our feeling that the upper bounds for $\alpha(r_o)$ and $\beta(r_o)$ that we have obtained are substantially above the true values of $\alpha(r_o)$ and $\beta(r_o)$.

It seems appropriate at this point to give a numerical example. Consider the problem of testing $H_o: \theta = \theta_o$ against $H_1: \theta = \theta_1$ with $\alpha = \beta = .05$ and $k = \theta_o/\theta_1 = 3$. One can readily verify from Table 5, that a non-sequential life test requires $r = 10$. The hypothesis that $\theta = \theta_o$ is accepted if $V(X_{10,n}) > \theta_o \chi^2_{.95}(20)/2 = 5.426\theta_o$, and rejected otherwise. Now let us see what happens in a sequential life test for the four truncation number of failures, $r_o = 10, 15, 20, 30$.

If $r_o = 10$, then

$$\alpha(10) \leq \alpha + \Pr\{10 \log 3 - \log 19 < \chi^2(20) < 10 \log 3\}$$

$$= \alpha + \Pr\{8.05 < \chi^2(20) < 10.99\} \simeq .05 + .04 = .09$$

and

$$\beta(10) \leq \beta + \Pr\{30 \log 3 \leq \chi^2(20) < 3\,(10 \log 3 - \log 19)\}$$

$$= \beta + \Pr\{32.96 \leq \chi^2(20) < 41.79\} \cong .05 + .04 = .09 .$$

If $r_o = 15$, then

$$\alpha(15) \leq \alpha + \Pr\{15 \log 3 - \log 19 < \chi^2(30) < 15 \log 3\}$$

$$= \alpha + \Pr\{13.54 < \chi^2(30) < 16.48\} \cong .05 + .025 = .075$$

and

$$\beta(15) \leq \beta + \Pr\left\{45 \log 3 \leq \chi^2(30) < 3(15 \log 3 + \log 19)\right\}$$

$$= \beta + \Pr\left\{49.44 \leq \chi^2(30) < 58.26\right\} \cong .05 + .015 = .065 \quad .$$

If $r_o = 20$, then

$$\alpha(20) \leq \alpha + \Pr(20 \log 3 - \log 19 < \chi^2(40) < 20 \log 3)$$

$$= \alpha + \Pr(19.04 < \chi^2(40) < 21.98) \cong .06$$

and

$$\beta(20) \leq \beta + \Pr\left\{60 \log 3 \leq \chi^2(40) < 3(20 \log 3 + \log 19)\right\}$$

$$= \beta + \Pr\left\{65.92 \leq \chi^2(40) < 74.76\right\} \cong .055 \quad .$$

If $r_o = 30$, then

$$\alpha(30) \leq \alpha + \Pr(30 \log 3 - \log 19 < \chi^2(60) < 30 \log 3)$$

$$= \alpha + \Pr(30.04 < \chi^2(60) < 32.96) \cong .051$$

$$\beta(30) \leq \beta + \Pr\left\{(90 \log 3 \leq \chi^2(60) < 3(30 \log 3 + \log 19)\right\}$$

$$= \beta + \Pr(98.88 \leq \chi^2(60) < 107.70) \cong .051 \quad .$$

Thus we see in this example that if we truncate the sequential life test at $r_o = 30$, i.e., at 3 times 10 , the r required for the non-sequential life test, then $\alpha(r_o)$ is approximately equal to α and $\beta(r_o)$ is approximately equal to β. Tables are being calculated for other values of α, β and θ_o/θ_1 and the indications are that what we observed in the example holds more generally. That is, truncation of the sequential life test at three times the number of failures required in the non-sequential life test will have virtually no effect on either α or β.

3. Truncation on the total observed life.

We have up to this point truncated the sequential life test by setting a definite limit on r_o, the number of items failed. We now wish to truncate the sequential life test by placing a definite limit V_o on the total observed life. We can achieve this by truncating the sequential life test at $V(t) = V_o$, i.e., by giving a new rule for the acceptance or rejection of $H_o: \theta = \theta_o$ when $V(t) = V_o$ if the sequential life test did not lead to a decision for $V(t) \leq V_o$. A simple and reasonable truncation rule at total life V_o is the following: If the sequential probability ratio test given by (31) does not lead to a final decision for $V(t) \leq V_o$, accept $H_o: \theta = \theta_o$ at total life V_o if

(2G.16) $$\log B < r \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o \leq 0 .$$

and reject H_o (accept $H_1: \theta = \theta_1$) if

(2G.17) $$0 < r \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o < \log A .$$

Truncating the sequential life test at total life V_o will change the Type I and Type II errors. They will no longer be α and β, the Type I and Type II errors, respectively, in the untruncated sequential case. The effect of the truncation on α and β depends, of course, on V_o.

The larger one makes V_o, the smaller are the truncation effects on α and β. Let us denote the resulting Type I and Type II errors as $\alpha(V_o)$ and $\beta(V_o)$, respectively, if the sequential life test is truncated at total life $V(t) = V_o$ at the latest. We now derive upper bounds for $\alpha(V_o)$ and $\beta(V_o)$.

To obtain an upper bound for $\alpha(V_o)$ we have to consider the cases in which the truncated sequential life test leads to the rejection of $H_o: \theta = \theta_o$ while the non-truncated sequential life test leads to the acceptance of H_o. Supposing that H_o holds (i.e., that $\theta = \theta_o$), let $\rho_o(V_o)$ be the probability that the sample random function associated with a life test is such that the truncated life test leads to rejection of H_o, while the non-truncated life test leads to the acceptance of H_o. Clearly we get

(2G.18)
$$\alpha(V_o) \leq \alpha + \rho_o(V_o) .$$

The reason for the inequality rather than the equality is that there may be sample random functions associated with life tests for which the truncated life test leads to acceptance of H_o, while the non-truncated life test leads to the rejection of H_o. Assuming that H_o is true, $\rho_o(V_o)$ is the probability that the random function associated with a life test is such that the following three conditions hold simultaneously:

(i) $\log B < r \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(t) < \log A$, for $V(t) < V_o$

(2G.19) (ii) $0 < r \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o < \log A$ and

(iii) When the sequential life test is continued beyond total life V_o, it terminates with the acceptance of H_o.

Assuming that H_o is true, let $\bar{\rho}_o(V_o)$ be the probability that condition (ii) holds, i.e.,

$$\text{(2G.20)} \qquad \bar{\rho}_o(V_o) = \Pr\left\{0 < r \log\frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o < \log A \,\middle|\, \theta = \theta_o\right\} .$$

Since the probability that (ii) is fulfilled cannot be smaller than the probability that conditions (i) , (ii) , and (iii) are fulfilled simultaneously, we have

$$\text{(2G.21)} \qquad \bar{\rho}_o(V_o) \geq \rho_o(V_o)$$

and, therefore,

$$\text{(2G.22)} \qquad \alpha(V_o) \leq \alpha + \bar{\rho}_o(V_o) .$$

Thus $\alpha + \bar{\rho}_o(V_o)$ is an upper bound for $\alpha(V_o)$. We show further on that $\bar{\rho}_o(V_o)$ can be computed easily.

To obtain an upper bound for $\beta(V_o)$, let us assume that $H_1: \theta = \theta_1$, is true and let $\rho_1(V_o)$ then be the probability that the truncated life test leads to the acceptance of H_o , while the non-truncated life test leads to the rejection of H_o . In other words, $\rho_1(V_o)$ is the probability (assuming that $\theta = \theta_1$ is true) that the sample random function associated with a life test is such that the following three conditions hold simultaneously:

$$\text{(2G.23)} \quad \begin{aligned} &\text{(i)} \quad \log B < r \log\frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(t) < \log A \text{ , for } V(t) < V_o \\ &\text{(ii)} \quad \log B < r \log\frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o \leq 0 \end{aligned}$$

(2G.23)

and

(iii) If the sequential life test is continued beyond total life V_o, it terminates with the acceptance of $H_1: \theta = \theta_1$.

(2G.24) Clearly $$\beta(V_o) \leq \beta + \rho_1(V_o) \quad .$$

Since it is difficult to determine $\rho_1(V_o)$ we give a simple upper bound first. Assuming that H_1 is true, let $\bar{\rho}_1(V_o)$ be the probability that condition (ii) holds, i.e.,

(2G.25) $$\bar{\rho}_1(V_o) = \Pr \left\{\log B < r \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o \leq 0 \,\middle|\, \theta = \theta_1\right\} \quad .$$

Then

(2G.26) $$\bar{\rho}_1(V_o) \geq \rho_1(V_o)$$

and hence

(2G.27) $$\beta(V_o) \leq \beta + \bar{\rho}_1(V_o) \quad .$$

We now show how to compute $\bar{\rho}_o(V_o)$ and $\bar{\rho}_1(V_o)$. To compute $\bar{\rho}_o(V_o)$, we recall that if V_o is preassigned, then under the hypothesis that $H_o: \theta = \theta_o$ is true, we are observing a Poisson process with rate parameter $\lambda_o = 1/\theta_o$ for a length of time V_o. Consequently,

(2G.28) $$\bar{\rho}_o(V_o) = \Pr \left\{ \frac{(k-1)\frac{V_o}{\theta_o}}{\log k} < r < \frac{(k-1)\frac{V_o}{\theta_o} + \log A}{\log k} \right\} = \sum_{m_o < r < n_o} p\left(r; \frac{V_o}{\theta_o}\right) ,$$

where $k = \frac{\theta_0}{\theta_1}$, $m_0 = (k-1)\frac{V_0}{\theta_0} \Big/ \log k$ and $n_0 = \left[(k-1)\frac{V_0}{\theta_0} + \log A\right] \Big/ \log k$.

In a similar way one can compute $\bar{\rho}_1(V_0)$. If V_0 is preassigned, then under the hypothesis that H_1: $\theta = \theta_1$ is true, we are observing a Poisson process with rate parameter $\lambda_1 = 1/\theta_1$ for a length of time V_0 . Consequently,

$$(2G.29) \qquad \bar{\rho}_1(V_0) = \Pr\left\{\frac{(k-1)\frac{V_0}{\theta_0} + \log B}{\log k} < r < \frac{(k-1)\frac{V_0}{\theta_0}}{\log k}\right\} = \sum_{m_1 < r < n_1} p(r; \frac{V_0}{\theta_1}) ,$$

where $m_1 = \left[(k-1)\frac{V_0}{\theta_0} + \log B\right] \Big/ \log k$ and $n_1 = (k-1)\frac{V_0}{\theta_0} \Big/ \log k$.

Thus we can summarize our results as follows:

$$(2G.30) \qquad \alpha(V_0) \leq \alpha + \sum_{m_0 < r < n_0} p(r; \frac{V_0}{\theta_0})$$

and

$$(2G.31) \qquad \beta(V_0) \leq \beta + \sum_{m_1 < r < n_1} p(r; \frac{V_0}{\theta_1})$$

where m_0 , n_0 , m_1 , n_1 are defined above. We are quite sure that the upper bounds for $\alpha(V_0)$ and $\beta(V_0)$ that we have obtained are substantially above the true values of $\alpha(V_0)$ and $\beta(V_0)$.

We now give a numerical example to illustrate what we have just discussed. Consider the problem of testing H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$, with $\alpha = \beta = .05$ and $k = \theta_0/\theta_1 = 3$. Let us see what happens for truncation times V_0 such that $V_0/\theta_0 = 3$, 6 , 9 , 12 , and 15 .

If $V_0/\theta_0 = 3$, then

$$\alpha(V_o) \leq \alpha + \sum_{m_o < r < n_o} p(r;\ 3) \quad , \text{ where}$$

$$m_o = (k-1)\frac{V_o}{\theta_o} \Big/ \log k = 6/1.10 = 5.45$$

and

$$n_o = \left[(k-1)\frac{V_o}{\theta_o} + \log A\right] \Big/ \log k = \frac{6 + 2.94}{1.10} = \frac{8.94}{1.10} = 8.13 \quad .$$

Hence

$$\alpha(V_o) \leq .05 + \sum_{r=6}^{8} p(r;\ 3) = .05 + .08 = .13 \quad .$$

Similarly

$$\beta(V_o) \leq \beta + \sum_{m_1 < r < n_1} p(r;\ 9) \quad , \quad \text{where}$$

$$m_1 = \left[(k-1)\frac{V_o}{\theta_o} + \log B\right] \Big/ \log k = \left[6 - 2.94\right] \Big/ 1.10$$

$$= 3.06/1.10 = 2.78$$

and

$$n_1 = (k-1)\frac{V_o}{\theta_o} \Big/ \log k = 5.45 \quad .$$

Hence

$$\beta(V_o) \leq .05 + \sum_{r=3}^{5} p(r;\ 9) = .16$$

If $\frac{V_o}{\theta_o} = 6$, then

$$\alpha(V_o) \leq \alpha + \sum_{r=11}^{13} p(r;\ 6) = .05 + .039 = .089$$

$$\beta(V_o) \leq \beta + \sum_{r=9}^{10} p(r;\ 18) = .05 + .023 = .073 \quad .$$

If $\frac{V_o}{\theta_o} = 9$, then

$$\alpha(V_o) \leq \alpha + \sum_{r=17}^{19} p(r;\, 9) = .060$$

and

$$\beta(V_o) \leq \beta + \sum_{r=14}^{16} p(r;\, 27) = .064 \quad .$$

If $\frac{V_o}{\theta_o} = 12$, then

$$\alpha(V_o) \leq \alpha + \sum_{r=22}^{24} p(r;\, 12) = .055$$

and

$$\beta(V_o) \leq \beta + \sum_{r=20}^{21} p(r;\, 36) = .054 \quad .$$

If $\frac{V_o}{\theta_o} = 15$, then

$$\alpha(V_o) \leq \alpha + \sum_{r=28}^{29} p(r;\, 15) = .052$$

and

$$\beta(V_o) \leq \beta + \sum_{r=25}^{27} p(r;\, 45) = .052 \quad .$$

Thus we see in this example that if we truncate the sequential life test at $V_o = 15\,\theta_o$, then $\alpha(V_o)$ and $\beta(V_o)$ are approximately equal to α and

β, respectively. For the non-sequential life test, truncation occurs at $V = 5.426\theta_o$. Thus in this example truncating the sequential life test at three times the V required for the non-sequential test has virtually no effect on either α or β. Tables are being calculated for other values of α, β, and θ_o/θ_1 and there are indications that what we observed in the example holds more generally.

Truncation on the number of items failed and total life.

Now it may happen that we would like to truncate the life test both with respect to the number of failures r_o and total life V_o. We first note that our truncated sequential life tests considered up to now are of this kind. Indeed, suppose that one truncates at $r = r_o$. Then we assert that this induces a truncation on total life,

$$V_o = \frac{r_o \log \frac{\theta_o}{\theta_1}}{\frac{1}{\theta_1} - \frac{1}{\theta_o}} = s r_o \quad . \qquad \text{[See Figure (5).]}$$

Thus if the random function representing the life test is such that one attains total life V_o (reaches BC) with fewer than r_o failures, then one knows that if one continues the test until r_o failures occur then the sample random function must cross either BD or DC and in either case we would accept H_o. Hence if one attains total life $V_o = s r_o$ with fewer than r_o failures, one can stop with acceptance of H_o. Similarly, if one truncates at total life $V = V_o$, then we assert that this induces a truncation on the number of failures

$$r_o = V_o\left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right)/\log \frac{\theta_o}{\theta_1} = V_o/s \quad . \qquad \text{[See Figure (6).]}$$

Thus if the random function representing the life test is such that the r_o'th failure occurs before total life V_o (i.e., reaches CG), then one knows that if one continues the test until total life V_o , then the sample random function must cross either GH or CH and in either case we would reject H_o . Hence if the r_o'th failure occurs before total life V_o one can stop with the rejection of H_o .

Suppose now that one preassigns both the number of failures r_o and total life V_o and truncates the sequential life test at $V = V_o$ and $r = r_o$. Then from the foregoing one can impose the following equivalent truncations:

(1) For r truncate at $r^* = \min \left[r_o , V_o\left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) \Big/ \log \frac{\theta_o}{\theta_1} \right] = \min \left[r_o, V_o/s \right]$

(2) For V truncate at $V^* = \min \left[V_o , \dfrac{r_o \log \frac{\theta_o}{\theta_1}}{\frac{1}{\theta_1} - \frac{1}{\theta_o}} \right] = \min \left[V_o, s\, r_o \right]$.

Clearly r^* and V^* will meet the condition

$$(3) \qquad r^* = V^*\left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) \Big/ \log \frac{\theta_o}{\theta_1} = V^*/s \ .$$

[See Figure (7).]

The truncation rule is as follows: if the sequential probability ratio test given by (31) does not lead to a decision for $V(t) \leq V^*$ and $r \leq r^*$ (i.e., if neither AB nor FG are intersected or crossed) , then if the sample random function associated with the life test hits the boundary $V = V^*$ (reaches BC) before reaching $r = r^*$ (CG) , accept H_o . If, however, the sample random function hits the boundary $r = r^*$ (CG) before reaching $V = V^*$ (BC) , reject H_o (accept H_1) .

Truncating the sequential life test at failure number r^* and at total life V^*, i.e., accepting H_o, if the sample random function associated with the life test meets AB or BC before crossing FG or meeting CG, and rejecting H_o if the sample random function crosses FG or meets CG before meeting AB or BC, will change the Type I and Type II erros. Let $\alpha(r^*, V^*)$ and $\beta(r^*, V^*)$ be the Type I and Type II errors, respectively, associated with the truncated test. It is clear from what we have said above that O.C. curves associated with truncating at $r = r^*$, $V = V^*$, where $V^* = sr^*$ coincide with those based on truncation at $r = r^*$ or $V = V^*$. Consequently $\alpha(r^*,V^*) = \alpha(r^*)$ and $\beta(r^*, V^*) = \beta(V^*)$. Upper bounds given previously for $\alpha(r^*)$ and $\beta(V^*)$ are automatically upper bounds for $\alpha(r^*, V^*)$ and $\beta(r^*,V^*)$.

Appendix 2H

Probability of termination of the sequential life test at preassigned values of r_o and V_o.

It is interesting to ask the question: what is the probability that the sequential life test will terminate with a number of failures less than or equal to some preassigned number, r_o, or after total life less than or equal to some preassigned value V_o ? Using considerations analogous to those in Wald's book on Sequential Analysis, pp. 58-60, we can state the following results which give lower bounds for the probability that the sequential procedure will terminate with a number of failures $r \leq r_o$ for the two values $\theta = \theta_o$ and $\theta = \theta_1$.

Consider the question of evaluating the probability that the sequential life test terminates with a number of failures $\leq r_o$. Then using considerations like those in Wald, we can assert that

$$(2H.1)\quad Pr(r \leq r_o \mid \theta = \theta_o) \geq Pr\left(r_o \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(x_{r_o}) \leq \log B \mid \theta = \theta_o\right)$$

$$= Pr\left[\chi^2 (2r_o) \geq \frac{2}{k-1} (r_o \log k - \log B) \right] ,$$

since $2\, V(x_{r_o}) \Big/ \theta_o$ is distributed as $\chi^2 (2r_o)$ under H_o.

And similarly

$$(2H.2)\quad Pr(r \leq r_o \mid \theta = \theta_1) \geq Pr\left[r_o \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V(x_{r_o}) \geq \log A \mid \theta = \theta_1\right]$$

$$= Pr\left(\chi^2 (2r_o) \leq \frac{2k}{k-1} (r_o \log k - \log A)\right)$$

since $2\,V(x_{r_o})/\theta_1$ is distributed as $\chi^2\,(2r_o)$ under H_1.

In a similar way we can evaluate the probability that the sequential life test terminates at total life $V(t) \leq V_o$. We can give the following lower bounds for the probability that the sequential procedure will terminate at total life $V(t) \leq V_o$ for the two values $\theta = \theta_o$ and $\theta = \theta_1$:

$$\text{(2H.3)}\quad \Pr(V(t) \leq V_o \mid \theta = \theta_o) \geq \Pr\left(r \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o \leq \log B \mid \theta = \theta_o\right)$$

$$= \sum_{0 \leq r \leq m_o} p\left(r; \frac{V_o}{\theta_o}\right), \text{ where } m_o = \left[(k-1)\frac{V_o}{\theta_o} + \log B\right] \Big/ \log k \;.$$

Since under the hypothesis that $H_o: \theta = \theta_o$ is true, we are observing a Poisson process with rate parameter $\lambda_o = 1/\theta_o$ for a length of time V_o. Also

$$\text{(2H.4)}\quad \Pr(V(t) \leq V_o \mid \theta = \theta_1) \geq \Pr\left(r \log \frac{\theta_o}{\theta_1} - \left(\frac{1}{\theta_1} - \frac{1}{\theta_o}\right) V_o \geq \log A \mid \theta = \theta_1\right)$$

$$= \sum_{r \geq m_1} p\left(r; \frac{V_o}{\theta_1}\right), \text{ where } m_1 = \left[(k-1)\frac{V_o}{\theta_o} + \log A\right] \Big/ \log k \;,$$

since under the hypothesis that $H_1: \theta = \theta_1$ is true, we are observing a Poisson process with rate parameter $\lambda_1 = 1/\theta_1$ for a length of time V_o.

We now give a numerical example to illustrate what we have just discussed. Consider the problem of testing $H_o: \theta = \theta_o$ against $H_1: \theta = \theta_1$ with $\alpha = \beta = .05$ and $k = \theta_o/\theta_1 = 3$. Let us compute lower bounds for $\Pr(r \leq r_o \mid \theta = \theta_o)$ and $\Pr(r \leq r_o \mid \theta = \theta_1)$ for $r_o = 10, 15, 20, 25, 30$ and lower bounds for $\Pr(V(t) \leq V_o \mid \theta = \theta_1)$ for $V_o/\theta_o = 3, 6, 9, 12$, and 15. We first compute $\Pr(r \leq r_o \mid \theta_i)$, $i = 1, 2$ for this example:

If $r_o = 10$, then

$$Pr(r \leq 10 \mid \theta = \theta_o) \geq Pr(\chi^2 (20) \geq 10.99 + 2.94)$$

$$= Pr(\chi^2 (20) \geq 13.93) = .83$$

and

$$Pr(r \leq 10 \mid \theta = \theta_1) \geq Pr(\chi^2 (20) \leq 3(10.99 - 2.94))$$

$$= Pr(\chi^2 (20) \leq 24.15) = .77 \ .$$

If $r_o = 15$, then

$$Pr(r \leq 15 \mid \theta = \theta_o) \geq Pr(\chi^2 (30) \geq 19.43) = .94$$

and

$$Pr(r \leq 15 \mid \theta = \theta_1) \geq Pr(\chi^2 (30) \leq 40.65) = .90 \ .$$

If $r_o = 20$, then

$$Pr(r \leq 20 \mid \theta = \theta_o) \geq Pr(\chi^2 (40) \geq 24.92) = .97$$

and

$$Pr(r \leq 20 \mid \theta = \theta_1) \geq Pr(\chi^2 (40) \leq 57.12) = .96 \ .$$

If $r_o = 25$, then

$$Pr(r \leq 25 \mid \theta = \theta_o) \geq Pr(\chi^2 (50) \geq 30.42) = .99$$

and

$$Pr(r \leq 25 \mid \theta = \theta_1) \geq Pr(\chi^2 (50) \leq 73.62) = .98 \ .$$

If $r_o = 30$, then

$$Pr(r \leq 30 \mid \theta = \theta_o) \geq Pr(\chi^2\ (60) \geq 35.91\ ) = .995$$

and

$$Pr(r \leq 30 \mid \theta = \theta_1) \geq Pr(\chi^2\ (60) \leq 90.15\ ) = .992\ .$$

Similarly we compute $Pr(V(t) \leq V_o \mid \theta = \theta_i)$, $i = 1,2$, for this example.

If $\frac{V_o}{\theta_o} = 3$, then

$$Pr(V(t) \leq V_o \mid \theta = \theta_o) \geq \sum_{0 \leq r \leq m_o} p(r;\ 3)\ , \text{ where}$$

$$m_o = \left[(k-1)\frac{V_o}{\theta_o} + \log B\right] \Big/ \log k = \left[6 - 2.94\right] \Big/ 1.1 = 2.8\ .$$

Hence

$$Pr(V(t) \leq V_o \mid \theta = \theta_o) \geq \sum_{r=0}^{2} p(r;\ 3) = .42$$

and

$$Pr(V(t) \leq V_o \mid \theta = \theta_1) \geq \sum_{r \geq m_1} p(r;\ 9)\ , \quad \text{where}$$

$$m_1 = \left[(k-1)\frac{V_o}{\theta_o} + \log A\right] \Big/ \log k = \left[6 + 2.94\right] \Big/ 1.1 = 8.1\ .$$

Hence $\quad Pr(V(t) \leq V_o \mid \theta = \theta_1) \geq \sum_{r=8}^{\infty} p(r;\ 9) = .55\ .$

If $\frac{V_o}{\theta_o} = 6$, then

$$\Pr(V(t) \leq V_o \mid \theta = \theta_o) \geq \sum_{r=0}^{8} p(r;\ 6) = .85$$

and

$$\Pr(V(t) \leq V_o \mid \theta = \theta_1) \geq \sum_{r=13}^{\infty} p(r;18) = .908 \ .$$

If $\frac{V_o}{\theta_o} = 9$, then

$$\Pr(V(t) \leq V_o \mid \theta = \theta_o) \geq \sum_{r=0}^{13} p(r;\ 9) = .926$$

and

$$\Pr(V(t) \leq V_o \mid \theta = \theta_1) \geq \sum_{r=19}^{\infty} p(r;\ 27) = .956 \ .$$

If $\frac{V_o}{\theta_o} = 12$, then

$$\Pr(V(t) \leq V_o \mid \theta = \theta_o) \geq \sum_{r=0}^{19} p(r;\ 12) = .979$$

and

$$\Pr(V(t) \leq V_o \mid \theta = \theta_1) \geq \sum_{r=24}^{\infty} p(r;\ 36) = .986 \ .$$

If $\frac{V_o}{\theta_o} = 15$, then

$$\Pr(V(t) \leq V_o \mid \theta = \theta_o) \geq \sum_{r=0}^{24} p(r;\ 15) = .989$$

and

$$\Pr(V(t) \leq V_o \mid \theta = \theta_1) \geq \sum_{r=29}^{\infty} p(r;\ 45) = .996 \ .$$

Tables for the probability that a sequential life test will terminate by a preassigned r_o and V_o are being computed for other values of α, β and $\frac{\theta_o}{\theta_1}$.

Appendix 2I

Upper and lower bounds for $L(\theta)$ and $E_\theta(r)$.

The formulae for $L(\theta)$ and $E_\theta(r)$, given by (37) and (38), respectively, are approximations to the actual $L(\theta)$ and actual $E_\theta(r)$ arising from the use of the sequential rule specified by the inequalities (31). The question arises as to how good these approximations are. A modification of the results of Wald on bounds for the O.C. and ASN curves in the binomial case and of results of Herbach on the discrete Poisson yields the following bounds on the actual $L(\theta)$ and $E_\theta(r)$:

(i) $$\frac{A^h - 1}{A^h - B^h} \leq L(\theta) \leq \frac{(kA)^h - 1}{(kA)^h - B^h} \; , \qquad h \neq 0 \text{ (that is, for } \theta \neq s) \; ,$$

(ii) $$\frac{L(\theta)\log B + [1 - L(\theta)]\log A}{\log k - \theta(1/\theta_1 - 1/\theta_o)} \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} E_\theta(r) \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} \frac{L(\theta)\log B + [1 - L(\theta)][\log A + \log k]}{\log k - \theta(1/\theta_1 - 1/\theta_o)}$$

where the upper inequality signs hold for $\theta < s$ and the lower inequality signs hold for $\theta > s$.

One unpleasant feature of the bounds given in (ii) is that they involve $L(\theta)$, which is unknown. However, this matters little in actual practice because the limits on $L(\theta)$ given by (i) are quite close together for the range of values of k and (α,β) covered in Table 10. Thus, for example, for

$k = \theta_o/\theta_1 = 3$, $\alpha = \beta = .05$, $A = (1 - \beta)/\alpha = 19$, $B = \beta/(1 - \alpha) = 1/19$,

we get that $.95 \leq L(\theta_o) \leq .983$ and $.05 \leq L(\theta_1) \leq .052$. The upper and lower bounds for $E_\theta(r)$ given by (ii) are close together for $\theta = \theta_0$ and comparatively far apart for $\theta = \theta_1$. Thus for the case $k = 3$ and $\alpha = \beta = .05$, the difference between the upper and lower bounds is $\leq .06$ for $\theta = \theta_0$ and is about 2.5 for $\theta = \theta_1$.

The left side of (ii) is the approximate formula (38) for $E_\theta(r)$ except that the $L(\theta)$ in (ii) refers to the exact value and the $L(\theta)$ in (38) is given by the approximation (37). In view of the preceding paragraph, the values of $E_{\theta_0}(r)$ given in Table 10 are very close to the correct values, while the values of $E_{\theta_1}(r)$ are essentially lower bounds for the correct value. We cannot say more unless we go through more extensive calculations of the sort to be described in Appendix 2J.

Appendix 2J

Some exact calculations of $L(\theta)$ and $E_\theta(r)$

Wald pointed out that in order to have a test of exactly strength (α, β) , the A and B in (31) should be replaced by A^* and B^* , where $A^* \leq A = (1 - \beta)/\alpha$ and $B^* \geq B = \beta/(1 - \alpha)$. In the present case, with information available continuously in time, $B^* = B = \beta/(1 - \alpha)$ since the acceptance of H_0 involves no excess over the boundary. However, acceptance of H_1 does, in general, entail a positive excess over the boundary, and all we can say initially about A^* is that it should lie between $A\,\theta_1/\theta_0$ and A . Thus using $A = (1 - \beta)/\alpha$ instead of A^* is an approximation.

The approximate test based on using A and B is suitable for all practical purposes, since one consequence of the inequalities in Appendix 2I is that the strength (α' , β') is such that $\alpha' \leq \alpha$, $\beta' \leq \beta/(1 - \alpha)$. Since α and β are generally small $(\leq .10$ say) a procedure based on A and B

provides essentially the same protection against errors of the first and second kind as does the test based on using A^* and B^*. However, the use of A rather than A^* in (31) will entail a small increase in $E_\theta(r)$, particularly for $\theta < s$.

As a practical matter, one would usually be content with a test based on (31) which uses A and B. As a matter of fact, this is what is done all the time by people faced with a practical decision problem. For most sequential problems, the problem of finding the A^* and B^* which will give exactly strength (α, β) has not been solved. One has to rely, in such cases, on the results of Wald which indicate that the errors involved in using A, B, and approximate formulae for $L(\theta)$ and $E_\theta(r)$ are "reasonably" small.

In the problem at hand we know, in view of the continuous availability of information, that $B^* = B = \beta/(1-\alpha)$. Furthermore, formulae are available for computing A^* and for computing O.C. and ASN curves exactly. The formulae for accomplishing these tasks are available in papers by Burman and by Dvoretzky, Kiefer, and Wolfowitz. While the computational labor involved in any special case is exceedingly heavy, the results of such computations do throw some light on how exact O.C. and ASN curves compare with those computed by using approximations.

Formulae (4.17) and (4.23) in the Dvoretzky, Kiefer, and Wolfowitz paper (similar formulae are given in Burman's paper, p. 102) were used to compute

(i) the exact O.C. and $E_\theta(r)$ curves for the sequential rule (31) with $B = \beta/(1-\alpha)$ and $A = (1-\beta)/\alpha$. This was done for the case $k = \theta_0/\theta_1 = 3$ and $\alpha = \beta = .05$, and

(ii) A^* (where $A\,\theta_1/\theta_0 \leq A^* \leq A$) such that the decision rule

$$\beta/(1-\alpha) < (\theta_0/\theta_1)^r \exp[-(1/\theta_1 - 1/\theta_0)\,V(t)] < A^*$$

has an O.C. curve for which $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) = \beta$ exactly, and then to compute $E_\theta(r)$ for the (B, A^*) rule. This was done for the cases $\alpha = \beta = .05$ and $k = \theta_0/\theta_1 = 3/2, 2$, and 3, and also for $\alpha = \beta = .01$ and $k = 3$.

The result of (i) was

$$L(\theta_0) = .968, \quad L(s) = .529, \quad L(\theta_1) = .051,$$

$$E_{\theta_0}(r) = 3.03, \quad E_s(r) = 8.10, \quad E_{\theta_1}(r) = 7.00.$$

Computation (ii) gave

$\alpha = \beta$	k	A^*	$E_{\theta_0}(r)$	$E_s(r)$	$E_{\theta_1}(r)$
.05	3	13.25	2.94	7.22	6.21
	2	15.1	8.64	18.0	13.8
	3/2	16.6	27.9	52.8	36.8
.01	3	68.9	5.00	17.5	10.5

Bearing in mind that the computations were carried through only in a small number of cases, one can make three observations:

(a) For the case $k = 3$ and $\alpha = \beta = .05$, the use of $B = 1/19$ and $A = 19$ results in getting $\alpha' = .032$ and $\beta' = .051$ as compared with $\alpha = \beta = .05$ when one uses $B^* = B = 1/19$ and $A^* = 13.25$. Also, $E_\theta(r)$ is increased by .09, .88, and .79 at $\theta = \theta_0, s, \theta_1$ respectively.

(b) Of more interest is the fact that the exact values of $E_\theta(r)$ for the (B,A^*) rule practically coincide with the approximate values of $E_\theta(r)$ computed for the (B,A) rule by formulae (38) and (39) and given in Table 10.

(c) In the range of values of $k = \theta_0/\theta_1$ and of α and β covered by Table 10, a good guess at the value of A^* is the value A^{**} lying midway between A and A/k, the upper and lower limits on A^*. This means that $A^{**} = (k+1)\,A/2k$. On the basis of our limited calculations we conjecture that in the range of values covered in Table 1, a sequential decision rule based on (31) with A replaced by A^{**} will have almost exactly strength (α, β). The values of $E_\theta(r)$ associated with a (B,A^{**}) rule will be given to a close approximation by (38).

Appendix 2K

An approximate formula for $E_\theta(t)$ in the nonreplacement case

A useful approximation to $E_\theta(t)$ in the nonreplacement case is given by $E_\theta(t) \sim \theta \log (n/[n - E_\theta(r)])$. This approximation is obtained by replacing $E_\theta(X_{k,n})$ in (42) by its approximation $\theta \log (n/[n - k]\theta)$. Thus (42) becomes (43)

$$E_\theta(t) \sim \theta\, E_\theta \left[\log \left(\frac{n}{n-k}\right)\right] \sim \theta \log \left(\frac{n}{n - E_\theta(r)}\right) .$$

This approximation has been tested numerically by calculations on truncated nonreplacement decision procedures, where the exact values of $E_\theta(t)$ can be computed and compared with the suggested approximation. The agreement is close.

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TABLE 1

Values of $\chi^2_{1-\alpha}(2r)/2r$

Number of failures

r \ α	.01	.05	.10	.25	.50
1	.010	.052	.106	.288	.693
2	.074	.178	.266	.481	.839
3	.145	.272	.367	.576	.891
4	.206	.342	.436	.634	.918
5	.256	.394	.486	.674	.934
6	.298	.435	.525	.703	.945
7	.333	.469	.556	.726	.953
8	.363	.498	.582	.744	.959
9	.390	.522	.604	.760	.963
10	.413	.543	.622	.773	.967
15	.498	.616	.687	.816	.978
20	.554	.663	.726	.842	.983
25	.594	.695	.754	.859	.987
30	.625	.720	.774	.872	.989
40	.669	.755	.803	.889	.992
50	.701	.779	.824	.901	.993
75	.751	.818	.855	.920	.996
100	.782	.841	.874	.931	.997

Note: This table is used in the following way:

Accept $\theta = \theta_0$ if $\hat{\theta}_{r,n} > \theta_0 \chi^2_{1-\alpha}(2r)/2r$ and reject otherwise.

Suppose, for example, that we want to discontinue a life test after $r = 3$ failures have occurred and that we want the life test to be such that a lot having mean life $\theta_0 = 1000$ hours is accepted with probability .90. Using formula (7) and table (1) the region of acceptance is given by

$$\hat{\theta}_{3,n} > (1000)(.367) = 367.$$

In words, one places n items on life test and stops testing after 3 items have failed. One then computes $\hat{\theta}_{3,n}$, an estimate of the mean life after 3 failures, using formula 2 in the non-replacement case and formula 3 in the replacement case. One accepts the lot if $\hat{\theta} > 367$ and rejects otherwise. Suppose, for example, that we place $n = 10$ items on test, do not replace items as they fail, and that the first 3 failure times are 50, 125, 250. In this case $\hat{\theta}_{3,10} = \frac{50+125+250+7(250)}{3} = \frac{2175}{3} = 725$. Since $725 > 367$, we accept the lot.

TABLE 2(a)

Values of θ accepted with probability p, when $\theta = 1$ is accepted with probability .99. Rule of action is to accept if $\hat{\theta}_{r,n} > \chi^2_{.99}(2r)/2r$

Number of failures r \ p	.99	.95	.90	.75	.50	.25	.10	.05	.01
1	1.0	.194	.095	.035	.014	.007	.004	.003	.002
2	1.0	.418	.279	.154	.088	.055	.038	.031	.022
3	1.0	.533	.396	.252	.163	.111	.082	.069	.052
4	1.0	.602	.472	.325	.224	.161	.123	.106	.082
5	1.0	.649	.526	.380	.274	.204	.160	.140	.110
6	1.0	.683	.566	.423	.315	.241	.193	.170	.136
7	1.0	.709	.598	.458	.349	.272	.221	.197	.160
8	1.0	.730	.624	.488	.379	.300	.247	.221	.182
9	1.0	.747	.646	.513	.405	.325	.270	.243	.202
10	1.0	.761	.664	.535	.427	.347	.291	.263	.220
15	1.0	.809	.726	.611	.510	.430	.371	.342	.294
20	1.0	.836	.763	.658	.563	.486	.428	.398	.348
25	1.0	.855	.788	.692	.602	.527	.470	.440	.390
30	1.0	.868	.807	.717	.632	.560	.504	.474	.424
40	1.0	.887	.833	.753	.675	.608	.554	.526	.477
50	1.0	.899	.851	.777	.705	.642	.591	.563	.516
75	1.0	.918	.878	.817	.754	.699	.653	.627	.583
100	1.0	.930	.895	.840	.785	.734	.692	.669	.627

Note: Tables 2(a) through 2(e) give O.C. curves associated with test procedures of the form: Accept if $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$. Time units are chosen in such a way that the probability of accepting $\theta = 1$ is $1 - \alpha$.

Examples of use of table 2a.

From table (1) we know that the acceptance rule $\hat{\theta}_{5,n} > 1000\ (.256) = 256$ will lead to the acceptance of a lot with mean life $\theta = 1000$, with probability .99. From table 2(a) we can say that a lot with mean life 526 will be accepted with probability .90, that a lot with mean life 274 will be accepted with probability .50, and a lot with mean life 160 will be accepted with probability .10.

TABLE 2(b)

Values of θ accepted with probability p, when $\theta = 1$ is accepted with probability .95. Rule of action is to accept if $\hat{\theta}_{r,n} > \chi^2_{.95}(2r)/2r$.

Number of failures

r \ p	.99	.95	.90	.75	.50	.25	.10	.05	.01
1	5.150	1.0	.488	.179	.074	.037	.022	.017	.011
2	2.394	1.0	.668	.370	.212	.132	.091	.075-	.054
3	1.875	1.0	.742	.473	.306	.209	.154	.130	.097
4	1.660	1.0	.783	.539	.372	.267	.205-	.176	.136
5	1.540	1.0	.810	.585	.422	.314	.246	.215	.170
6	1.463	1.0	.829	.619	.461	.352	.282	.249	.199
7	1.410	1.0	.844	.646	.493	.384	.312	.277	.225
8	1.370	1.0	.855	.668	.519	.411	.338	.303	.249
9	1.339	1.0	.864	.687	.542	.435	.361	.325	.270
10	1.314	1.0	.872	.702	.561	.455	.382	.345	.289
15	1.237	1.0	.898	.755	.630	.531	.459	.422	.363
20	1.196	1.0	.912	.788	.674	.581	.512	.475	.416
25	1.170	1.0	.922	.810	.705	.617	.550	.515	.456
30	1.152	1.0	.930	.826	.728	.645	.581	.546	.489
40	1.128	1.0	.940	.849	.761	.685	.625	.593	.538
50	1.112	1.0	.946	.865	.785	.714	.658	.627	.574
75	1.089	1.0	.956	.889	.822	.761	.711	.683	.635
100	1.076	1.0	.962	.904	.844	.790	.745	.719	.675

Example: If the life test is discontinued after $r = 1$ failure occurs, and if a lot with mean life $\theta = 1$ is accepted with probability .95, then a lot with mean life $\theta^* = .074$ is accepted with probability .50. Rule of action is: Accept if $\hat{\theta}_{1,n} > .052$.

If the life test is discontinued after $r = 5$ failures occur, and if a lot with mean life $\theta = 1$ is accepted with probability .95, then a lot with mean life $\theta = .422$ is accepted with probability .50. Rule of action is: Accept if $\hat{\theta}_{5,n} > .394$.

TABLE 2(c)

Values of θ accepted with probability p, when $\theta = 1$ is accepted with probability .90. Rule of action is to accept if $\hat{\theta}_{r,n} > \chi^2_{.90}(2r)/2r$.

Number of failures r \ p	.99	.95	.90	.75	.50	.25	.10	.05	.01
1	10.550	2.049	1.0	.367	.152	.076	.046	.035	.023
2	3.582	1.496	1.0	.553	.317	.198	.137	.112	.080
3	2.528	1.348	1.0	.638	.412	.281	.207	.175	.131
4	2.120	1.277	1.0	.688	.475	.342	.261	.225	.174
5	1.902	1.235	1.0	.722	.521	.388	.304	.266	.210
6	1.765	1.206	1.0	.747	.556	.425	.340	.300	.240
7	1.672	1.186	1.0	.766	.584	.455	.370	.329	.267
8	1.602	1.170	1.0	.782	.607	.481	.396	.354	.291
9	1.549	1.157	1.0	.795	.627	.503	.418	.376	.312
10	1.506	1.147	1.0	.805	.643	.522	.438	.396	.331
15	1.377	1.114	1.0	.842	.702	.592	.512	.471	.405
20	1.311	1.096	1.0	.863	.739	.637	.561	.521	.456
25	1.269	1.084	1.0	.878	.764	.669	.597	.558	.495
30	1.239	1.076	1.0	.888	.783	.694	.624	.587	.526
40	1.201	1.064	1.0	.903	.810	.729	.666	.631	.572
50	1.175	1.057	1.0	.914	.829	.755	.695	.662	.606
75	1.139	1.046	1.0	.930	.859	.795	.743	.714	.664
100	1.118	1.039	1.0	.939	.877	.820	.774	.747	.701

Example: If the life test is discontinued after $r = 10$ failures occur, and if a lot with mean life $\theta = 1$ is accepted with probability .90, then a lot with mean life $\theta^* = .643$ is accepted with probability .50, and a lot with mean life $\theta^{**} = .396$ is accepted with probability .05. Rule of action is: Accept if $\hat{\theta}_{10,n} > .622$.

TABLE 2(d)

Values of θ accepted with probability p, when $\theta = 1$ is accepted with probability .75. Rule of action is to accept if $\hat{\theta}_{r,n} > \chi^2_{.75}(2r)/2r$.

Number of failures

r \ p	.99	.95	.90	.75	.50	.25	.10	.05	.01
1	28.750	5.582	2.725	1.0	.415	.207	.125	.090	.062
2	6.475	2.705	1.807	1.0	.573	.357	.247	.203	.145
3	3.902	2.113	1.568	1.0	.646	.441	.325	.274	.206
4	3.081	1.855	1.453	1.0	.690	.496	.380	.327	.252
5	2.634	1.710	1.385	1.0	.721	.537	.421	.308	.290
6	2.303	1.615	1.339	1.0	.744	.568	.455	.401	.322
7	2.181	1.547	1.305	1.0	.762	.594	.483	.429	.349
8	2.050	1.496	1.279	1.0	.777	.615	.506	.453	.372
9	1.949	1.456	1.259	1.0	.789	.633	.526	.474	.393
10	1.871	1.424	1.242	1.0	.799	.648	.544	.492	.411
15	1.637	1.324	1.188	1.0	.834	.703	.608	.559	.481
20	1.519	1.270	1.159	1.0	.856	.738	.650	.604	.528
25	1.446	1.235	1.139	1.0	.870	.762	.680	.636	.564
30	1.395	1.211	1.126	1.0	.881	.781	.703	.661	.592
40	1.329	1.178	1.107	1.0	.897	.807	.737	.698	.633
50	1.286	1.157	1.094	1.0	.907	.826	.761	.725	.664
75	1.225	1.125	1.076	1.0	.924	.855	.800	.768	.714
100	1.190	1.106	1.065	1.0	.934	.874	.824	.796	.746

Example: If the life test is discontinued after $r = 10$ failures occur, and if a lot with mean life $\theta = 1$ is accepted with probability .75, then a lot with mean life $\theta^{*} = 1.424$ is accepted with probability .95 and a lot with mean life $\theta^{**} = .544$ is accepted with probability .10. Rule of action is: Accept if $\hat{\theta}_{10,n} > .773$.

TABLE 2(e)

Values of θ accepted with probability p, when $\theta = 1$ is accepted with probability .50. Rule of action is to accept if $\hat{\theta}_{r,n} > \chi^2_{.50}(2r)/2r$.

Number of failures

r \ p	.99	.95	.90	.75	.50	.25	.10	.05	.01
1	69.300	13.456	6.569	2.410	1.0	.500	.301	.231	.150
2	11.303	4.722	3.155	1.746	1.0	.623	.432	.354	.253
3	6.133	3.271	2.426	1.548	1.0	.682	.502	.425	.318
4	4.462	2.687	2.104	1.448	1.0	.719	.550	.474	.366
5	3.652	2.371	1.920	1.387	1.0	.744	.584	.510	.403
6	3.176	2.170	1.799	1.344	1.0	.764	.611	.539	.433
7	2.862	2.030	1.712	1.312	1.0	.779	.633	.563	.458
8	2.639	1.926	1.647	1.288	1.0	.792	.652	.583	.479
9	2.472	1.846	1.596	1.268	1.0	.802	.667	.601	.498
10	2.341	1.782	1.554	1.251	1.0	.812	.681	.616	.515
15	1.962	1.586	1.424	1.198	1.0	.843	.729	.670	.576
20	1.775	1.484	1.354	1.169	1.0	.862	.759	.705	.618
25	1.661	1.419	1.309	1.149	1.0	.876	.781	.731	.648
30	1.583	1.374	1.277	1.135	1.0	.886	.798	.750	.671
40	1.482	1.314	1.234	1.115	1.0	.900	.821	.779	.706
50	1.418	1.275	1.206	1.102	1.0	.910	.838	.799	.731
75	1.325	1.217	1.164	1.082	1.0	.926	.865	.832	.773
100	1.274	1.185	1.140	1.071	1.0	.935	.882	.852	.799

Example: If the life test is discontinued after $r = 10$ failures occur, and if a lot with mean life $\theta = 1$ is accepted with probability .50, then a lot with mean life $\theta^* = 1.782$ is accepted with probability .95 and a lot with mean life $\theta^{**} = .681$ is accepted with probability .10. Rule of action is: Accept if $\hat{\theta}_{10,n} > .967$.

TABLE 3(a)

Values of $E(X_{r,n})/\theta$, where $X_{r,n}$ is the r^{th} Smallest Value in a Random Sample of Size n Drawn from a Distribution whose Probability Density Function is $\frac{1}{\theta} e^{-x/\theta}$, x, $\theta > 0$.

In the table $r = 1(1)n$ and $n = 1(1)20(5)30(10)100$.

r \ n	1	2	3	4	5	6
1	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667
2		1.5000	0.8333	0.5833	0.4500	0.3667
3			1.8333	1.0833	0.7833	0.6167
4				2.0833	1.2833	0.9500
5					2.2833	1.4500
6						2.4500

r \ n	7	8	9	10	11	12
1	0.1429	0.1250	0.1111	0.1000	0.0909	0.0833
2	0.3095	0.2679	0.2361	0.2111	0.1909	0.1742
3	0.5095	0.4345	0.3790	0.3361	0.3020	0.2742
4	0.7595	0.6345	0.5456	0.4790	0.4270	0.3853
5	1.0929	0.8845	0.7456	0.6456	0.5699	0.5104
6	1.5929	1.2179	0.9956	0.8456	0.7365	0.6532
7	2.5929	1.7179	1.3290	1.0956	0.9365	0.8199
8		2.7179	1.8290	1.4290	1.1865	1.0199
9			2.8290	1.9290	1.5199	1.2699
10				2.9290	2.0199	1.6032
11					3.0199	2.1032
12						3.1032

TABLE 3a (Con't.)

r/n	13	14	15	16	17	18
1	0.0769	0.0714	0.0667	0.0625	0.0588	0.0556
2	0.1603	0.1484	0.1381	0.1292	0.1213	0.1144
3	0.2512	0.2317	0.2150	0.2006	0.1880	0.1769
4	0.3512	0.3226	0.2984	0.2775	0.2594	0.2435
5	0.4623	0.4226	0.3893	0.3609	0.3363	0.3150
6	0.5873	0.5337	0.4893	0.4518	0.4197	0.3919
7	0.7301	0.6587	0.6004	0.5518	0.5106	0.4752
8	0.8968	0.8016	0.7254	0.6629	0.6106	0.5661
9	1.0968	0.9682	0.8682	0.7879	0.7217	0.6661
10	1.3468	1.1682	1.0349	0.9307	0.8467	0.7773
11	1.6801	1.4182	1.2349	1.0974	0.9896	0.9023
12	2.1801	1.7516	1.4849	1.2974	1.1562	1.0451
13	3.1801	2.2516	1.8182	1.5474	1.3562	1.2118
14		3.2516	2.3182	1.8807	1.6062	1.4118
15			3.3182	2.3807	1.9396	1.6618
16				2.3807	2.4396	1.9951
17					3.4396	2.4951
18						3.4951

r	n = 19				
1	0.0526	7	0.4445	14	1.2644
2	0.1082	8	0.5279	15	1.4644
3	0.1670	9	0.6188	16	1.7144
4	0.2295	10	0.7188	17	2.0477
5	0.2962	11	0.8299	18	2.5477
6	0.3676	12	0.9549	19	3.5477
		13	1.0977		

TABLE 3a (Con't.)

n = 20

r		r		r		r	
1	0.0500	6	0.3462	11	0.7687	16	1.5144
2	0.1026	7	0.4176	12	0.8799	17	1.7644
3	0.1582	8	0.4945	13	1.0049	18	2.0977
4	0.2170	9	0.5779	14	1.1477	19	2.5977
5	0.2795	10	0.6688	15	1.3144	20	3.5977

n = 25

r		r		r		r	
1	0.0400	8	0.3764	14	0.7961	20	1.5326
2	0.0817	9	0.4352	15	0.8870	21	1.7326
3	0.1251	10	0.4977	16	0.9870	22	1.9826
4	0.1706	11	0.5644	17	1.0981	23	2.3160
5	0.2182	12	0.6358	18	1.2231	24	2.8160
6	0.2682	13	0.7127	19	1.3660	25	3.8160
7	0.3209						

n = 30

r		r		r	
1	0.0333	11	0.4472	21	1.1660
2	0.0678	12	0.4999	22	1.2771
3	0.1035	13	0.5554	23	1.4021
4	0.1406	14	0.6143	24	1.5450
5	0.1790	15	0.6768	25	1.7117
6	0.2190	16	0.7434	26	1.9117
7	0.2607	17	0.8149	27	2.1617
8	0.3042	18	0.8918	28	2.4950
9	0.3496	19	0.9751	29	2.9950
10	0.3972	20	1.0660	30	3.9950

TABLE 3a (Cont.)

n = 40

r		r		r		r	
1	0.0250	11	0.3169	21	0.7308	31	1.4496
2	0.0506	12	0.3514	22	0.7834	32	1.5607
3	0.0770	13	0.3871	23	0.8390	33	1.6857
4	0.1040	14	0.4241	24	0.8978	34	1.8285
5	0.1318	15	0.4626	25	0.9603	35	1.9952
6	0.1603	16	0.5026	26	1.0270	36	2.1952
7	0.1897	17	0.5443	27	1.0984	37	2.4452
8	0.2200	18	0.5877	28	1.1753	38	2.7785
9	0.2513	19	0.6332	29	1.2587	39	3.2785
10	0.2836	20	0.6808	30	1.3496	40	4.2785

n = 50

r		r		r		r	
1	0.0200	16	0.3810	31	0.9515	46	2.4159
2	0.0404	17	0.4104	32	1.0041	47	2.6659
3	0.0612	18	0.4407	33	1.0597	48	2.9992
4	0.0825	19	0.4720	34	1.1185	49	3.4992
5	0.1043	20	0.5042	35	1.1810	50	4.4992
6	0.1265	21	0.5376	36	1.2476		
7	0.1492	22	0.5720	37	1.3191		
8	0.1725	23	0.6077	38	1.3960		
9	0.1963	24	0.6448	39	1.4793		
10	0.2207	25	0.6832	40	1.5702		
11	0.2457	26	0.7232	41	1.6702		
12	0.2713	27	0.7649	42	1.7813		
13	0.2976	28	0.8084	43	1.9063		
14	0.3246	29	0.8538	44	2.0492		
15	0.3524	30	0.9015	45	2.2159		

TABLE 3a (Con't.)

n = 60

r		r		r	
1	0.0167	21	0.4263	41	1.1321
2	0.0336	22	0.4520	42	1.1848
3	0.0509	23	0.4783	43	1.2403
4	0.0684	24	0.5053	44	1.2991
5	0.0863	25	0.5331	45	1.3616
6	0.1044	26	0.5617	46	1.4283
7	0.1230	27	0.5911	47	1.4997
8	0.1418	28	0.6214	48	1.5767
9	0.1611	29	0.6526	49	1.6600
10	0.1807	30	0.6849	50	1.7509
11	0.2007	31	0.7182	51	1.8509
12	0.2211	32	0.7527	52	1.9620
13	0.2419	33	0.7884	53	2.0870
14	0.2632	34	0.8255	54	2.2299
15	0.2849	35	0.8639	55	2.3965
16	0.3071	36	0.9039	56	2.5965
17	0.3299	37	0.9456	57	2.8465
18	0.3531	38	0.9891	58	3.1799
19	0.3769	39	1.0345	59	3.6799
20	0.4013	40	1.0821	60	4.6799

TABLE 3a (Con't.)

n = 70

r		r		r		r	
1	0.0143	19	0.3140	37	0.7440	55	1.5146
2	0.0288	20	0.3336	38	0.7743	56	1.5813
3	0.0435	21	0.3536	39	0.8056	57	1.6527
4	0.0584	22	0.3740	40	0.8378	58	1.7296
5	0.0736	23	0.3949	41	0.8712	59	1.8130
6	0.0889	24	0.4161	42	0.9057	60	1.9039
7	0.1046	25	0.4379	43	0.9414	61	2.0039
8	0.1204	26	0.4601	44	0.9784	62	2.1150
9	0.1366	27	0.4828	45	1.0169	63	2.2400
10	0.1530	28	0.5061	46	1.0569	64	2.3828
11	0.1696	29	0.5229	47	1.0985	65	2.5495
12	0.1866	30	0.5543	48	1.1420	66	2.7495
13	0.2038	31	0.5793	49	1.1875	67	2.9995
14	0.2214	32	0.6049	50	1.2351	68	3.3328
15	0.2392	33	0.6313	51	1.2851	69	3.8328
16	0.2574	34	0.6583	52	1.3377	70	4.8328
17	0.2759	35	0.6861	53	1.3933		
18	0.2948	36	0.7146	54	1.4521		

TABLE 3a (con't.)

n = 80

r		r		r		r	
1	0.0125	21	0.3023	41	0.7119	61	1.4177
2	0.0252	22	0.3192	42	0.7376	62	1.4704
3	0.0380	23	0.3365	43	0.7639	63	1.5259
4	0.0510	24	0.3540	44	0.7909	64	1.5848
5	0.0641	25	0.3719	45	0.8187	65	1.6473
6	0.0775	26	0.3900	46	0.8473	66	1.7139
7	0.0910	27	0.4086	47	0.8767	67	1.7853
8	0.1047	28	0.4274	48	0.9070	68	1.8623
9	0.1185	29	0.4467	49	0.9382	69	1.9456
10	0.1326	30	0.4663	50	0.9705	70	2.0365
11	0.1469	31	0.4863	51	1.0038	71	2.1365
12	0.1614	32	0.5067	52	1.0383	72	2.2476
13	0.1761	33	0.5275	53	1.0740	73	2.3726
14	0.1911	34	0.5488	54	1.1111	74	2.5155
15	0.2062	35	0.5705	55	1.1495	75	2.6821
16	0.2216	36	0.5928	56	1.1895	76	2.8821
17	0.2372	37	0.6155	57	1.2312	77	3.1321
18	0.2531	38	0.6387	58	1.2747	78	3.4655
19	0.2692	39	0.6625	59	1.3201	79	3.9655
20	0.2856	40	0.6869	60	1.3677	80	4.9655

TABLE 3a (con't.)

n = 90

r		r		r		r	
1	0.0111	26	0.3387	51	0.8290	76	1.8310
2	0.0223	27	0.3543	52	0.8547	77	1.9024
3	0.0337	28	0.3702	53	0.8810	78	1.9794
4	0.0452	29	0.3863	54	0.9080	79	2.0627
5	0.0568	30	0.4027	55	0.9358	80	2.1536
6	0.0686	31	0.4194	56	0.9644	81	2.2536
7	0.0805	32	0.4363	57	0.9938	82	2.3647
8	0.0926	33	0.4536	58	1.0241	83	2.4897
9	0.1047	34	0.4711	59	1.0553	84	2.6326
10	0.1171	35	0.4890	60	1.0876	85	2.7992
11	0.1296	36	0.5071	61	1.1209	86	2.9992
12	0.1422	37	0.5257	62	1.1554	87	3.2492
13	0.1551	38	0.5445	63	1.1911	88	3.5825
14	0.1681	39	0.5637	64	1.2282	89	4.0826
15	0.1812	40	0.5834	65	1.2666	90	5.0826
16	0.1945	41	0.6034	66	1.3066		
17	0.2081	42	0.6238	67	1.3483		
18	0.2218	43	0.6446	68	1.3918		
19	0.2356	44	0.6659	69	1.4372		
20	0.2497	45	0.6876	70	1.4848		
21	0.2640	46	0.7098	71	1.5348		
22	0.2785	47	0.7326	72	1.5875		
23	0.2932	48	0.7558	73	1.6430		
24	0.3081	49	0.7796	74	1.7018		
25	0.3233	50	0.8040	75	1.7643		

TABLE 3a (con't.)

n = 100

r		r		r		r	
1	0.0100	26	0.2994	51	0.7082	76	1.4114
2	0.0201	27	0.3129	52	0.7286	77	1.4531
3	0.0303	28	0.3266	53	0.7494	78	1.4966
4	0.0406	29	0.3405	54	0.7707	79	1.5420
5	0.0510	30	0.3545	55	0.7924	80	1.5896
6	0.0616	31	0.3688	56	0.8147	81	1.6396
7	0.0722	32	0.3833	57	0.8374	82	1.6923
8	0.0829	33	0.3980	58	0.8606	83	1.7478
9	0.0938	34	0.4130	59	0.8844	84	1.8066
10	0.1048	35	0.4281	60	0.9088	85	1.8691
11	0.1159	36	0.4435	61	0.9338	86	1.9358
12	0.1272	37	0.4591	62	0.9595	87	2.0072
13	0.1385	38	0.4750	63	0.9858	88	2.0842
14	0.1500	39	0.4911	64	1.0128	89	2.1675
15	0.1616	40	0.5075	65	1.0406	90	2.2584
16	0.1734	41	0.5242	66	1.0692	91	2.3584
17	0.1853	42	0.5411	67	1.0986	92	2.4695
18	0.1974	43	0.5584	68	1.1289	93	2.5945
19	0.2096	44	0.5759	69	1.1601	94	2.7374
20	0.2219	45	0.5938	70	1.1924	95	2.9040
21	0.2344	46	0.6119	71	1.2257	96	3.1040
22	0.2471	47	0.6305	72	1.2602	97	3.3540
23	0.2599	48	0.6493	73	1.2959	98	3.6874
24	0.2729	49	0.6686	74	1.3330	99	4.1874
25	0.2860	50	0.6882	75	1.3714	100	5.1874

Example: Find the expected waiting time for the 10th failure in a sample of size 20. Assume that θ = 1000 hours.

Solution: This means $E(X_{10,20})$ = (.6688) 1000 = 668.8 hours.

TABLE 3(b)

Values of $E(X_{r,n})/\theta$, where $X_{r,n}$ is the r^{th} smallest value in a random sample of size n drawn from a distribution whose probability density function is $f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x \geq 0$, $\theta > 0$. In the table $r = 1(1)10(5)30(10)50(25)100$ and $n = kr$, with $k = 1(1)10(10)20$.

r \ n	r	2r	3r	4r	5r	6r	7r	8r	9r	10r	20r
1	1.0	.5000	.3333	.2500	.2000	.1667	.1429	.1250	.1111	.1000	.0500
2	1.5000	.5833	.3667	.2679	.2111	.1742	.1484	.1292	.1144	.1026	.0506
3	1.8333	.6167	.3790	.2742	.2150	.1769	.1503	.1306	.1155	.1035	.0509
4	2.0833	.6345	.3854	.2775	.2170	.1782	.1512	.1313	.1161	.1040	.0510
5	2.2833	.6456	.3893	.2795	.2182	.1790	.1518	.1318	.1164	.1043	.0510
6	2.4500	.6532	.3919	.2809	.21[illegible]	.1796	.1522	.1321	.1166	.1044	.0511
7	2.5929	.6587	.3938	.2818	.2196	.1800	.1525	.1323	.1168	.1046	.0511
8	2.7179	.6629	.3952	.2825	.2200	.1803	.1527	.1324	.1169	.1047	.0511
9	2.8290	.6661	.3963	.2831	.2204	.1805	.1528	.1325	.1170	.1047	.0511
10	2.9290	.6688	.3972	.2836	.2207	.1807	.1530	.1326	.1171	.1048	.0512
15	3.3182	.6768	.4000	.2849	.2215	.1812	.1534	.1329	.1173	.1050	.0512
20	3.5977	.6808	.4013	.2856	.2219	.1815	.1536	.1331	.1174	.1051	.0512
25	3.8160	.6832	.4022	.2860	.2221	.1817	.1537	.1332	.1775	.1051	.0512
30	3.9950	.6849	.4027	.2863	.2223	.1818	.1538	.1332	.1176	.1052	.0512
40	4.2785	.6869	.4034	.2866	.2225	.1819	.1539	.1333	.1176	.1052	.0513
50	4.4992	.6882	.4038	.2869	.2226	.1820	.1539	.1334	.1176	.1052	.0513
75	4.9014	.6898	.4044	.2871	.2228	.1821	.1540	.1334	.1177	.1053	.0513
100	5.1874	.6907	.4046	.2873	.2229	.1822	.1540	.1334	.1177	.1053	.0513

Example: Compute the expected waiting time of the 15th failure in a sample of size 45. Assume that $\theta = 2000$ hours.

Solution: Using $r = 15$, $n = 3r$, we get $E(X_{r,n}) = (.4000)(2000) = 800$ hours.

TABLE 3(c)

Ratio of Expected Waiting Time to Observe the r^{th} Failure in Samples of Size n and r , Respectively.

$$\alpha_{r,n} = E(X_{r,n})/E(X_{r,r})$$

r \ n	1	2	3	4	5	10	15	20
1	1	.50	.33	.25	.20	.10	.067	.050
2		1	.56	.39	.30	.14	.092	.068
3			1	.59	.43	.18	.12	.087
4				1	.62	.23	.14	.104
5					1	.28	.18	.125
10						1	.35	.23

Example: Compare the expected waiting time to observe the 10th failure in a sample of size 20 with the expected waiting time to observe the 10th failure in a sample of size 10. The answer is $\alpha_{10,20} = .23$.

Table 4(a)

Values of $\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ for $\alpha = .01$.

r \ n	2r	3r	4r	5r	6r	7r	8r	9r	10r	20r
1	.005	.003	.003	.002	.002	.001	.001	.001	.001	.0005
2	.043	.027	.020	.016	.013	.011	.010	.008	.008	.004
3	.089	.055	.040	.031	.026	.022	.019	.017	.015	.007
4	.131	.079	.057	.045	.037	.031	.027	.024	.021	.011
5	.165	.100	.072	.056	.046	.039	.034	.030	.027	.013
6	.195	.117	.084	.065	.054	.045	.039	.035	.031	.015
7	.219	.131	.094	.073	.060	.051	.044	.039	.035	.017
8	.241	.143	.103	.080	.065	.055	.048	.042	.038	.019
9	.260	.155	.110	.086	.070	.060	.052	.046	.041	.020
10	.276	.164	.117	.091	.075	.063	.055	.048	.043	.021
15	.337	.199	.142	.110	.090	.076	.066	.058	.052	.025
20	.377	.222	.158	.123	.101	.085	.074	.065	.058	.028
25	.406	.239	.170	.132	.108	.091	.079	.070	.062	.030
30	.428	.252	.179	.139	.114	.096	.083	.074	.066	.032
40	.460	.270	.192	.149	.122	.103	.089	.079	.070	.034
50	.482	.283	.201	.156	.128	.108	.094	.082	.074	.036
75	.518	.304	.216	.167	.137	.116	.100	.088	.079	.039
100	.540	.316	.225	.174	.142	.120	.104	.092	.082	.040

Remark: Truncated non-replacement tests of the form accept if $X_{r,n} > T = \theta_0 \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ have virtually the same O.C. curve as tests of the form accept if $\hat{\theta}_{r,n} > C = \theta_0 \chi^2_{1-\alpha}(2r)/2r$. In the above table, α (the type I error) is .01 , when $\theta = \theta_0$.

Example: Consider a non-replacement situation, where we start with $n = 18$ items where $r = 6$, $\theta_0 = 1000$ hours and $\alpha = .01$. The test procedure is accept if $X_{6,18} > 117$. In words: Accept if the 6th failure has not yet occurred by 117 hours and reject if the 6th failure occurs before 117 hours have elapsed. Such a plan will accept a lot with mean life 1000 hours with probability .99.

Table 4(b)

Values of $\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ for $\alpha = .05$.

r \ n	2r	3r	4r	5r	6r	7r	8r	9r	10r	20r
1	.026	.017	.013	.010	.009	.007	.006	.006	.005	.003
2	.104	.065	.048	.038	.031	.026	.023	.020	.018	.009
3	.168	.103	.075	.058	.048	.041	.036	.031	.028	.014
4	.217	.132	.095	.074	.061	.052	.045	.040	.036	.017
5	.254	.153	.110	.086	.071	.060	.052	.046	.041	.020
6	.284	.170	.122	.095	.078	.066	.057	.051	.045	.022
7	.309	.185	.132	.103	.084	.072	.062	.055	.049	.024
8	.330	.197	.141	.110	.090	.076	.066	.058	.052	.025
9	.348	.207	.148	.115	.094	.080	.069	.061	.055	.027
10	.363	.216	.154	.120	.098	.083	.072	.064	.057	.028
15	.417	.246	.175	.136	.112	.094	.082	.072	.065	.032
20	.451	.266	.189	.147	.120	.102	.088	.078	.070	.034
25	.475	.280	.199	.154	.126	.107	.093	.082	.073	.036
30	.493	.290	.206	.160	.131	.111	.096	.085	.076	.037
40	.519	.305	.216	.168	.137	.116	.101	.089	.079	.039
50	.536	.315	.223	.173	.142	.120	.104	.092	.082	.040
75	.564	.331	.235	.182	.149	.126	.109	.096	.086	.042
100	.581	.340	.242	.187	.153	.130	.112	.099	.089	.043

Remark: Truncated tests of the form, accept if $X_{r,n} > T = \theta_o \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ have virtually the same O.C. curve as tests of the form, accept if $\hat{\theta}_{r,n} > C = \theta_o \chi^2_{1-\alpha}(2r)/2r$. In the above table, α (the Type I error) is .05, when $\theta = \theta_o$.

Example: For $n = 18$, $r = 6$, $\theta_o = 1000$ hours, and $\alpha = .05$, the acceptance region is: Accept if $X_{6,18} > 170$ hours.

Table 4(c)

Values of $\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ for $\alpha = .10$

r \ n	2r	3r	4r	5r	6r	7r	8r	9r	10r	20r
1	.053	.035	.026	.021	.018	.015	.013	.012	.011	.005
2	.155	.098	.071	.056	.046	.039	.034	.030	.027	.013
3	.226	.139	.101	.079	.065	.055	.048	.042	.038	.019
4	.277	.168	.121	.095	.078	.066	.057	.051	.045	.022
5	.314	.189	.136	.106	.087	.074	.064	.057	.051	.025
6	.343	.206	.147	.115	.094	.080	.069	.061	.055	.027
7	.366	.219	.157	.122	.100	.085	.074	.065	.058	.028
8	.386	.230	.164	.128	.105	.089	.077	.068	.061	.030
9	.402	.239	.171	.133	.109	.092	.080	.071	.063	.031
10	.416	.247	.176	.137	.112	.095	.082	.073	.065	.032
15	.465	.275	.196	.152	.124	.105	.091	.081	.072	.035
20	.494	.291	.207	.161	.132	.112	.097	.085	.076	.037
25	.515	.303	.216	.167	.137	.116	.100	.089	.079	.039
30	.530	.312	.222	.172	.141	.119	.103	.091	.081	.040
40	.552	.324	.230	.179	.146	.124	.107	.094	.084	.041
50	.567	.333	.236	.183	.150	.127	.110	.097	.087	.042
75	.590	.346	.245	.190	.156	.132	.114	.101	.090	.044
100	.604	.354	.251	.195	.159	.135	.117	.103	.092	.045

Remark: Truncated tests of the form, accept if $X_{r,n} > T = \theta_o \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ have virtually the same O.C. curve as tests of the form, accept if $\hat{\theta}_{r,n} > C = \theta_o \chi^2_{1-\alpha}(2r)/2r$. In the above table, α (the Type I error) is .10, when $\theta = \theta_o$.

Example: For $n = 18$, $r = 6$, $\theta_o = 1000$ hours, and $\alpha = .10$, the acceptance region is: Accept if $X_{6,18} > 206$ hours.

Table 4(d)

Values of $\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ for $\alpha = .25$

r \ n	2r	3r	4r	5r	6r	7r	8r	9r	10r	20r
1	.144	.096	.072	.058	.048	.041	.036	.032	.029	.014
2	.281	.176	.129	.102	.084	.071	.062	.055	.049	.024
3	.355	.218	.158	.124	.102	.087	.075	.067	.060	.029
4	.402	.244	.176	.138	.113	.096	.083	.074	.066	.032
5	.435	.262	.188	.147	.121	.102	.089	.078	.070	.034
6	.459	.276	.197	.154	.126	.107	.093	.082	.073	.036
7	.478	.286	.205	.159	.131	.111	.096	.085	.076	.037
8	.493	.294	.210	.164	.134	.114	.099	.087	.078	.038
9	.506	.301	.215	.168	.137	.116	.101	.089	.080	.039
10	.517	.307	.219	.171	.140	.118	.102	.091	.081	.040
15	.552	.326	.232	.181	.148	.125	.108	.096	.086	.042
20	.573	.338	.240	.187	.153	.129	.112	.099	.088	.043
25	.587	.345	.246	.191	.156	.132	.114	.101	.090	.044
30	.597	.351	.250	.194	.159	.134	.116	.103	.092	.045
40	.611	.359	.255	.198	.162	.137	.119	.105	.094	.046
50	.620	.364	.258	.201	.164	.139	.120	.106	.095	.046
75	.635	.372	.264	.205	.168	.142	.123	.108	.097	.047
100	.643	.377	.267	.208	.170	.143	.124	.110	.098	.048

Remark: Truncated tests of the form, accept if $X_{r,n} > T = \theta_0\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ have virtually the same O.C. curve as tests of the form, accept if $\hat{\theta}_{r,n} > C = \theta_0\chi^2_{1-\alpha}(2r)/2r$. In the above table, α (the Type I error) is .25, when $\theta = \theta_0$.

Example: For $n = 18$, $r = 6$, $\theta_0 = 1000$ hours, and $\alpha = .25$, the acceptance region is: Accept if $X_{6,18} > 276$ hours.

Table 4(e)

Values of $\chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ for $\alpha = .50$

r \ n	2r	3r	4r	5r	6r	7r	8r	9r	10r	20r
1	.346	.231	.173	.139	.116	.099	.087	.077	.069	.035
2	.489	.308	.225	.177	.146	.125	.108	.096	.086	.042
3	.549	.338	.244	.192	.158	.134	.116	.103	.092	.045
4	.582	.354	.255	.199	.164	.139	.121	.107	.095	.047
5	.603	.364	.261	.204	.167	.142	.123	.109	.097	.048
6	.617	.370	.265	.207	.170	.144	.125	.110	.099	.048
7	.628	.375	.269	.209	.172	.145	.126	.111	.100	.049
8	.636	.379	.271	.211	.173	.146	.127	.112	.100	.049
9	.641	.382	.273	.212	.174	.147	.128	.113	.101	.049
10	.647	.384	.274	.213	.175	.148	.128	.113	.101	.050
15	.662	.391	.279	.217	.177	.150	.130	.115	.103	.050
20	.669	.394	.281	.218	.178	.151	.131	.115	.103	.050
25	.674	.397	.282	.219	.179	.152	.131	.116	.104	.051
30	.677	.398	.283	.220	.180	.152	.132	.116	.104	.051
40	.681	.400	.284	.221	.180	.153	.132	.117	.104	.051
50	.683	.401	.285	.221	.181	.153	.132	.117	.104	.051
75	.687	.403	.286	.222	.181	.153	.133	.117	.105	.051
100	.689	.403	.286	.222	.182	.154	.133	.117	.105	.051

Remark: Truncated tests of the form, accept if $X_{r,n} > T = \theta_o \chi^2_{1-\alpha}(2r)E(X_{r,n})/2r$ have virtually the same O.C. curve as tests of the form, accept if $\hat{\theta}_{r,n} > C = \theta_o \chi^2_{1-\alpha}(2r)/2r$. In the above table, α (the Type I error) is .50, when $\theta = \theta_o$.

Example: For $n = 18$, $r = 6$, $\theta_o = 1000$ hours, and $\alpha = .50$, the acceptance region is: Accept if $X_{6,18} > 370$ hours.

Table 5

Values of r and $\chi^2_{1-\alpha}(2r)/2r$ such that the acceptance region $\hat{\theta}_{r,n} > \theta_o \chi^2_{1-\alpha}(2r)/2r$ is such that $L(\theta_o) = 1 - \alpha$ and $L(\theta_1) \leq \beta$.

θ_o/θ_1	r	$\chi^2_{1-\alpha}(2r)/2r$	r	$\chi^2_{1-\alpha}(2r)/2r$	r	$\chi^2_{1-\alpha}(2r)/2r$	r	$\chi^2_{1-\alpha}(2r(/2r$
	$\alpha = .01$, $\beta = .01$		$\alpha = .01$, $\beta = .05$		$\alpha = .01$, $\beta = .10$		$\alpha = .01$, $\beta = .25$	
3/2	136	.8114	101	.7831	83	.7625	60	.7244
2	46	.6892	35	.6492	30	.6247	22	.5715
3	19	.5445	15	.4985	13	.4692	10	.4130
5	9	.3897	8	.3633	7	.3329	5	.2558
10	5	.2558	4	.2058	4	.2058	3	.1453
	$\alpha = .05$, $\beta = .01$		$\alpha = .05$, $\beta = .05$		$\alpha = .05$, $\beta = .10$		$\alpha = .05$, $\beta = .25$	
3/2	95	.8374	67	.8079	55	.7890	35	.7391
2	33	.7319	23	.6834	19	.6548	13	.5915
3	13	.5915	10	.5426	8	.4976	6	.4355
5	7	.4694	5	.3940	4	.3416	3	.2725
10	4	.3416	3	.2725	3	.2725	2	.1778
	$\alpha = .10$, $\beta = .01$		$\alpha = .10$, $\beta = .05$		$\alpha = .10$, $\beta = .10$		$\alpha = .10$, $\beta = .25$	
3/2	77	.8570	52	.8269	41	.8058	25	.7538
2	26	.7583	18	.7123	15	.6866	9	.6036
3	11	.6383	8	.5820	6	.5253	4	.4363
5	5	.4865	4	.4363	3	.3673	3	.3673
10	3	.3673	2	.2660	2	.2660	2	.2660
	$\alpha = .25$, $\beta = .01$		$\alpha = .25$, $\beta = .05$		$\alpha = .25$, $\beta = .10$		$\alpha = .25$, $\beta = .25$	
3/2	52	.9033	32	.8758	23	.8526	12	.7932
2	17	.8275	11	.7836	8	.7445	5	.6737
3	7	.7261	5	.6737	4	.6339	2	.4808
5	3	.5758	2	.4808	2	.4808	1	.2875
10	2	.4808	2	.4808	1	.2875	1	.2875

Example: Find a life test which possesses the following O.C. curve: If the mean life is $\theta_0 = 900$ hours, it is accepted with probability .95; if the mean life is $\theta_1 = 300$ hours, it is accepted with probability $\leq .10$.

Solution: In this example $\theta_0/\theta_1 = 3$, $\alpha = .05$, and $\beta = .10$, therefore the required number of failures is $r = 8$. The region of acceptance is given by $\hat{\theta}_{8,n} > (900)(.4976) = 448$. In words: Stop life testing after 8 failures have occurred. If the mean life based on the 8 failures that have occurred > 448, accept; otherwise, reject.

Table 6

Values of n , the sample size, needed in truncated replacement procedures.

θ_0/θ_1 \ θ_0/T_0	3	5	10	20	3	5	10	20
	$\alpha = .01$	$\beta = .01$			$\alpha = .01$	$\beta = .05$		
3/2	331	551	1103	2207	237	395	790	1581
2	95	158	317	634	68	113	227	454
3	31	51	103	206	22	37	74	149
5	10	17	35	70	8	14	29	58
10	4*	6	12	25	3*	4	8	16
	$\alpha = .01$	$\beta = .10$			$\alpha = .01$	$\beta = .25$		
3/2	189	316	632	1265	130	217	434	869
2	56	93	187	374	37	62	125	251
3	18	30	60	121	12	20	41	82
5	7	11	23	46	4*	7*	13*	25
10	2	4	8	16	2*	2	4	8
	$\alpha = .05$	$\beta = .01$			$\alpha = .05$	$\beta = .05$		
3/2	238	397	795	1591	162	270	541	1082
2	72	120	241	483	47	78	157	314
3	23	38	76	153	16	27	54	108
5	9	16	32	65	6*	10*	19	39
10	4	6	13	27	3*	4	8	16
	$\alpha = .05$	$\beta = .10$			$\alpha = .05$	$\beta = .25$		
3/2	130	216	433	867	77	129	258	517
2	37	62	124	248	23	38	76	153
3	11	19	39	79	7	13	26	52
5	4	7*	13	27	3*	4	8	16
10	3*	4	8	16	1	2*	3	7
	$\alpha = .10$	$\beta = .01$			$\alpha = .10$	$\beta = .05$		
3/2	197	329	659	1319	128	214	429	859
2	59	98	197	394	38	64	128	256
3	21	35	70	140	13	23	46	93
5	7	12	24	48	5	8	17	34
10	3	5	11	22	2*	3*	5	10

Table 6 (cont'd)

θ_o/T_o / θ_o/θ_1	3	5	10	20	3	5	10	20
	$\alpha = .10$		$\beta = .10$		$\alpha = .10$		$\beta = .25$	
3/2	99	165	330	660	56	94	188	376
2	30	51	102	205	16	27	54	108
3	9	15	31	63	5	8	17	34
5	4*	6*	11	22	3	5	11	22
10	2*	2	5	10	1	2	5	10
	$\alpha = .25$		$\beta = .01$		$\alpha = .25$		$\beta = .05$	
3/2	140	234	469	939	84	140	280	560
2	42	70	140	281	25	43	86	172
3	15	25	50	101	10	16	33	67
5	5	8	17	34	3*	5*	10*	19
10	2	4	9	19	2	4	9	19
	$\alpha = .25$		$\beta = .10$		$\alpha = .25$		$\beta = .25$	
3/2	58	98	196	392	28	47	95	190
2	17	29	59	119	10	16	33	67
3	7	12	25	50	2	4	9	19
5	3*	4	9	19	1*	2*	3*	6*
10	1*	2*	3*	5	1*	1	2	5

***Remark**: It was indicated that if one uses the θ_o in table (5) and sets the sample size $n = \left[\frac{\theta_o}{T_o} \frac{\chi^2_{1-\alpha}(2r_o)}{2}\right]$, then it may happen that while $L(\theta_o) \geq 1-\alpha$, $L(\theta_1)$ may be slightly $> \beta$. One way of getting around this is to use $n+1$ items (rather than n items) and to use the slightly smaller truncation time $T_o'' = \theta_o \chi^2_{1-\alpha}(2r_o)/2(n+1)$. The test based on $\mathrm{Min}(X_{r_o,n+1};\ T_o'')$ will have $L(\theta_o) = 1-\alpha$ and $L(\theta_1) \leq \beta$. In the above table such an adjustment had to be made in the following cases:

Table 6 (cont'd)

α	β	θ_o/T_o	θ_o/θ_1	(n+1)	T''_o/T_o
.01	.01	3	10	4	.959
.01	.05	3	10	3	.823
.01	.25	3	5	4	.959
.01	.25	5	5	7	.914
.01	.25	10	5	13	.984
.01	.25	3	10	2	.654
.05	.05	3	5	6	.985
.05	.05	5	5	10	.985
.05	.05	3	10	3	.818
.05	.10	5	5	7	.976
.05	.10	3	10	3	.818
.05	.25	3	5	3	.818
.05	.25	5	10	2	.889
.10	.05	3	10	2	.798
.10	.05	5	10	3	.887
.10	.10	3	5	4	.827
.10	.10	5	5	6	.918
.10	.10	3	10	2	.798
.25	.05	3	5	3	.962
.25	.05	5	5	5	.962
.25	.05	10	5	10	.962
.25	.10	3	5	3	.962
.25	.10	3	10	1	.863
.25	.10	5	10	2	.719
.25	.10	10	10	3	.958
.25	.25	3	5	1	.863
.25	.25	5	5	2	.719
.25	.25	10	5	3	.958
.25	.25	20	5	6	.958
.25	.25	3	10	1	.863

Example: Find a truncated replacement plan for which T_o = 500 hours, which will accept a lot with mean life = 10,000 hours at least 90 percent of the time and reject a lot with mean life = 2,000 hours at least 90 percent of the time.

Solution: In this case θ_o = 10,000, θ_1 = 2,000, $\alpha = \beta = .10$. Since $\theta_o/\theta_1 = 5$, $\alpha = \beta = .10$, we see from Table 5 that the rejection number is $r_o = 3$. Corresponding to $\theta_o/\theta_1 = 5$, $\theta_o/T_o = 20$, $\alpha = \beta = .10$, one sees from Table 6 that the sample size is n=22.

Table 6 (cont'd)

Thus the derived truncated replacement plan meeting the requirements is as follows: Start the life test with $n=22$ items. As soon as an item fails replace it by a new item. Accept the lot if: $\min(X_{3,22};500) = 500$ (i.e., if 3 failures have not occurred by 500 hours, stop life testing and accept). Reject the lot if: $\min(X_{3,22};500) = X_{3,22}$ (i.e., if the 3rd failure runs before 500 hours, stop at the third failure and reject).

<u>Example</u>: Find a truncated replacement plan for which $T_o = 500$ hours, which will accept a lot with mean life = 1500 hours at least 95% of the time and reject a lot with mean life = 150 hours at least 95% of the time.

<u>Solution</u>: In this case $\theta_o = 1500$, $\theta_1 = 150$, $\alpha = \beta = .05$.

Since $\theta_o/\theta_1 = 10$, $\alpha = \beta = .05$, we see from Table 5 that the rejection number is $r_o = 3$. Corresponding to $\theta_o/\theta_1 = 10$, $\theta_o/T_o = 3$, $\alpha = \beta = .05$, one sees from Table 6 that the appropriate sample size to use is 3. Since this number has an asterisk (*) attached to it we see that we can actually use the smaller truncation time $T_o'' = .818T_o = (.818)500 = 409$. Thus the desired truncated replacement plan meeting the requirements is as follows:

Start the life test with 3 items. As soon as an item fails replace it by a new item.

Accept the lot if: $\text{Min}(X_{3,3};409) = 409$ (i.e., if 3 failures have not occurred by 409 hours, stop life testing and accept).

Reject the lot if: $\text{Min}(X_{3,3};409) = X_{3,3}$ (i.e., if the 3rd failure occurs before 409 hours, stop at the 3rd failure and reject).

Table 7

Values of n, the sample size, needed in truncated replacement procedures.

θ_o/T_o → / θ_o/θ_1 ↓	3	5	10	20	3	5	10	20
	$\alpha = .01$	$\beta = .01$			$\alpha = .01$	$\beta = .05$		
3/2	403	622	1172	2275	291	448	842	1632
2	119	182	340	657	87	132	245	472
3	41	61	113	216	30	45	82	157
5	15	22	39	74	13	18	33	62
10	6	9	15	28	4	6	10	18
	$\alpha = .01$	$\beta = .10$			$\alpha = .01$	$\beta = .25$		
3/2	234	359	675	1307	162	248	465	899
2	72	109	202	390	49	74	137	262
3	25	37	67	128	18	26	46	87
5	11	15	26	50	6	9	15	28
10	4	6	10	18	3	4	6	10
	$\alpha = .05$	$\beta = .01$			$\alpha = .05$	$\beta = .05$		
3/2	289	447	843	1639	198	305	575	1116
2	90	138	258	499	59	90	168	326
3	30	45	83	160	21	32	59	113
5	13	20	36	69	8	12	22	41
10	6	9	15	29	4	5	9	17
	$\alpha = .05$	$\beta = .10$			$\alpha = .05$	$\beta = .25$		
3/2	159	245	462	895	96	147	276	535
2	47	72	134	258	30	45	83	160
3	16	24	43	83	11	16	29	55
5	6	9	15	29	4	5	9	17
10	4	5	9	17	2	2	4	8
	$\alpha = .10$	$\beta = .01$			$\alpha = .10$	$\beta = .05$		
3/2	238	369	699	1358	136	242	456	886
2	73	112	210	407	48	73	137	265
3	27	40	75	145	18	27	50	97
5	10	14	26	51	7	10	19	36
10	5	7	12	23	2	3	6	11

Table 7 (cont'd)

θ_o/θ_1 \ θ_o/T_o	3	5	10	20	3	5	10	20
	$\alpha = .10$		$\beta = .10$		$\alpha = .10$		$\beta = .25$	
3/2	121	186	351	681	69	107	201	389
2	39	59	110	213	21	31	58	113
3	12	18	34	66	7	10	19	36
5	5	7	12	23	5	7	12	23
10	2	3	6	11	2	3	6	11
	$\alpha = .25$		$\beta = .01$		$\alpha = .25$		$\beta = .05$	
3/2	168	261	496	965	101	156	296	576
2	51	79	149	289	31	48	91	177
3	19	29	54	105	12	19	36	69
5	6	10	18	36	3	5	10	20
10	3	5	10	20	3	5	10	20
	$\alpha = .25$		$\beta = .10$		$\alpha = .25$		$\beta = .25$	
3/2	71	110	207	403	34	53	101	196
2	22	33	63	123	12	19	36	69
3	9	14	27	52	3	5	10	20
5	3	5	10	20	1	1	3	6
10	1	1	3	6	1	1	3	6

Example: Find a truncated non-replacement life test for which T_o=500 hours, which will accept a lot with mean life = 10,000 hours at least 90% of the time and reject a lot with mean life = 2,000 hours at least 90% of the time.

Solution: In this case θ_o=10,000, θ_1=2,000, $\alpha = \beta$ =.10. Since θ_o/θ_1=5, $\alpha = \beta$=.10, we find from Table 5 that the rejection number is r_o=3 . Corresponding to θ_o/θ_1=5, θ_o/T_o=20, $\alpha = \beta$ =.10, one sees from Table 7 that the sample size n=23 . Thus the derived truncated non-replacement plan meeting the requirements is as follows: Start the life test with n=23 items. Don't replace items as they fail. Accept the lot if $\min(X_{3,23};500)$= 500 (i.e., if 3 failures have not occurred by 500 hours, stop life testing and accept) Reject the lot if: $\min(X_{3,23};500)$=$X_{3,23}$ (i.e., if the 3rd failure occurs before 500 hours, stop at the 3rd failure and reject).

Table 8

Values of r_o (upper numbers) and of $\chi^2_{1-\alpha}(2r_o)/2$ (lower numbers) such that the test based on using a sampling plan with sample size equal to $[\chi^2_{1-\alpha}(2r_o)/2p_o]$ and with rejection number r_o will have an OC curve such that $L(p_o)=1-\alpha$ and $L(p_1) \leq \beta$. $L(p)$ is the probability of accepting a lot having fraction defective p.

p_1/p_o	$\alpha = .01$			$\alpha = .05$			$\alpha = .10$		
	$\beta = .01$	.05	.10	.01	.05	.10	.01	.05	.10
3/2	136 110.4	101 79.1	83 63.3	95 79.6	67 54.1	55 43.4	77 66.0	52 43.0	41 33.0
2	46 31.7	35 22.7	30 18.7	33 24.2	23 15.7	19 12.4	26 19.7	18 12.8	15 10.3
5/2	27 16.4	21 11.8	18 9.62	19 12.4	14 8.46	11 6.17	15 10.3	11 7.02	9 5.43
3	19 10.3	15 7.48	13 6.10	13 7.69	10 5.43	8 3.98	11 7.02	8 4.66	6 3.15
4	12 5.43	10 4.13	9 3.51	9 4.70	7 3.29	6 2.61	7 3.90	5 2.43	4 1.75
5	9 3.51	8 2.91	7 2.33	7 3.29	5 1.97	4 1.37	5 2.43	4 1.75	3 1.10
10	5 1.28	4 .823	4 .823	4 1.37	3 .818	3 .818	3 1.10	2 .532	2 .532

Example: Find a life test having the following properties; I accept at least 90% of the lots for which the probability of failing before some time T_o is $\leq .01$ and will reject at least 95% of the lots for which the probability of failing before $T_o \geq .10$.

Solution: In this problem, $p_o = .01$, $p_1 = .10$, $\alpha = .10$ and $\beta = .05$. Thus $p_1/p_o = 10$ and so we see from Table 8 that $r_o = 2$ and $N = [.532/.01] = 53$. Thus the life test is as follows: Place 53 items on test. If 2 or more failures occur before time T_o, reject. If one or fewer failures occur before time T_o, accept.

Table 9

Values of h_o, h_1, and s for various values of α, β, and θ_1. The normalized value, $\theta_o=1$, is used.

α	β	$\theta_1=2/3$ h_o	$\theta_1=2/3$ h_1	$\theta_1=1/2$ h_o	$\theta_1=1/2$ h_1	$\theta_1=1/3$ h_o	$\theta_1=1/3$ h_1
.01	.01	9.1902	9.1902	4.5951	4.5951	2.2976	2.2976
.01	.05	5.9714	9.1078	2.9857	4.5539	1.4928	2.2769
.05	.01	9.1078	5.9714	4.5539	2.9857	2.2769	1.4928
.05	.05	5.8889	5.8889	2.9444	2.9444	1.4722	1.4722

θ_1	s
2/3	.8109
1/2	.6931
1/3	.5493

Example: Find a sequential life test for the case when $\alpha=.05$, $\beta=.05$, $\theta_o=300$ hours and $\theta_1=100$ hours.

Solution: For this case, $h_o=h_1=1.4722$ (since $\theta_1=\frac{1}{3}$ if θ_o is normalized as 1), and $s=.5493$.

Therefore the region (35) is given by:

$$300[-1.4722+.5493r] < V(t) < 300[1.4722+.5493r] \quad .$$

Simplifying this gives:

$$-442+165r < V(t) < 442+165r \quad .$$

The life test is continued so long as $V(t)$, the total observed life up to time t, satisfies both inequalities. As soon as the inequalities are violated , one accepts H_o (i.e., $\theta_o=300$) if $V(t) > 442 + 165r$ and one rejects H_o (i.e., accepts H_1 ($\theta_1=100$) if $V(t) < -442 + 165r$.

Table 10

Approximate values of $E_\theta(r)$ for sequential tests for various values of $k=\theta_0/\theta_1$, α and β

$k=\theta_0/\theta_1$		3/2		2		3	
α		.01	.05	.01	.05	.01	.05
θ	β						
0	.01	12	8	7	5	5	3
	.05	12	8	7	5	5	3
θ_1	.01	62.4	40.3	23.3	15.1	10.4	6.74
	.05	60.4	36.7	22.6	13.7	10.1	6.14
s	.01	128	82.7	43.9	28.3	17.5	11.3
	.05	82.7	52.7	28.3	18.0	11.3	7.18
θ_0	.01	47.6	44.2	14.7	13.6	5.00	4.63
	.05	30.8	28.0	9.48	8.64	3.23	2.94
∞	any	0	0	0	0	0	0

Example: Find $E_{\theta_0}(r)$ if one is testing $\theta_0/\theta_1 = 3$ with $\alpha = .05$, and $\beta = .05$.

Solution: The expected number of items failed in reaching a decision when $\theta=\theta_0$ is $E_{\theta_0}(r) = 2.94$.

Figure 1(a$_1$

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .99 .

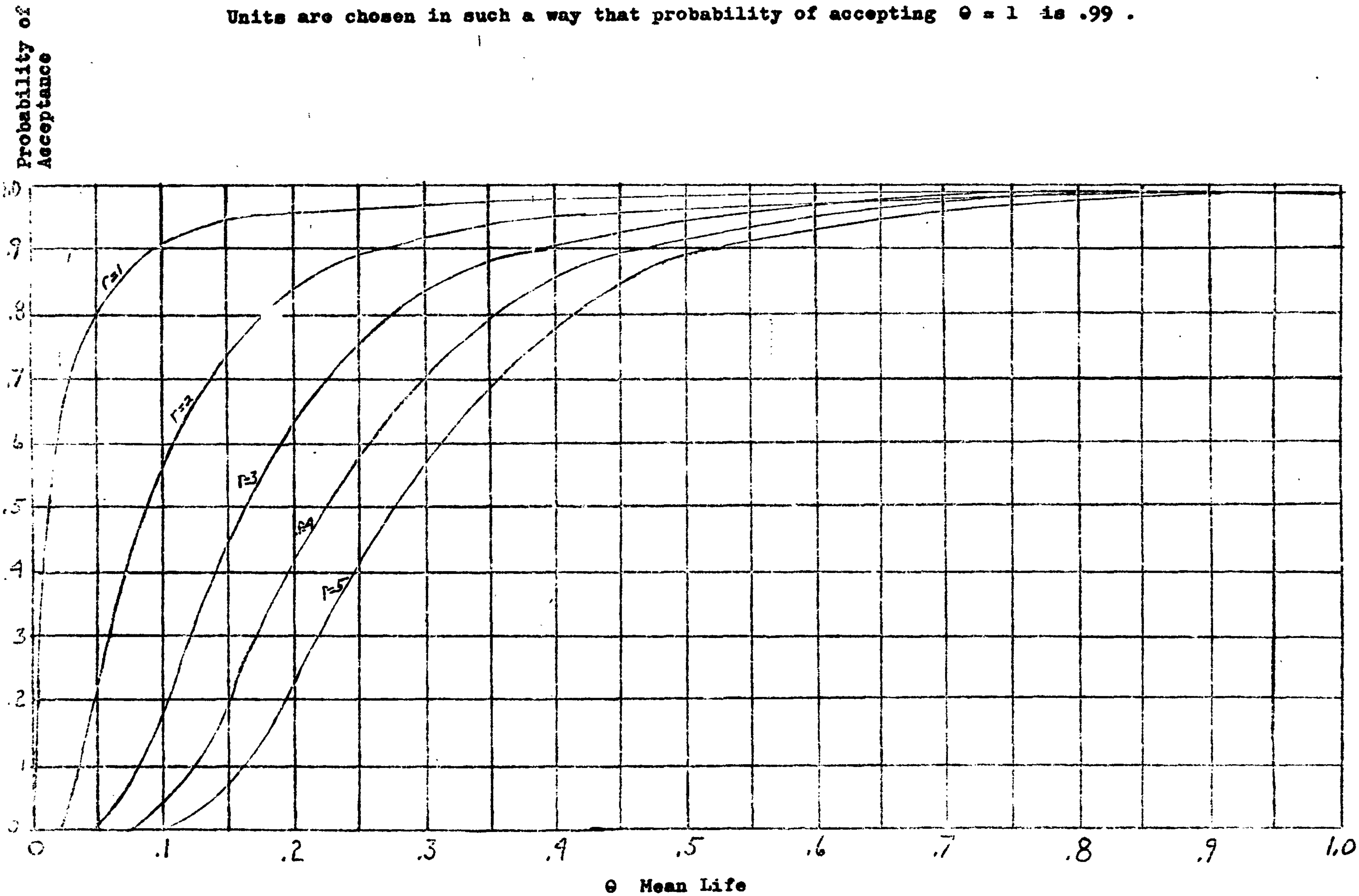

Figure 1 $_2$)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .99 .

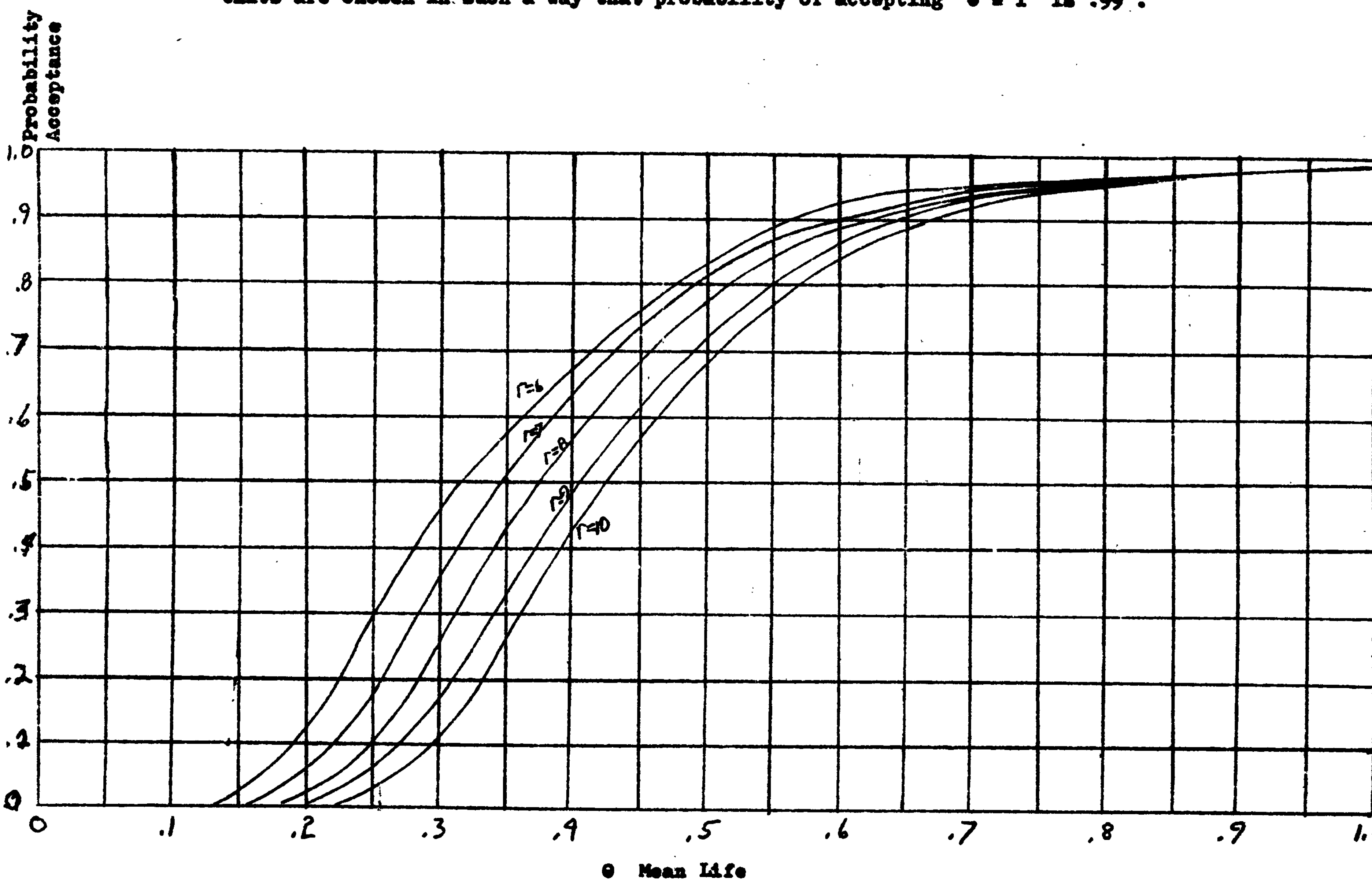

Figure 1(a_3)

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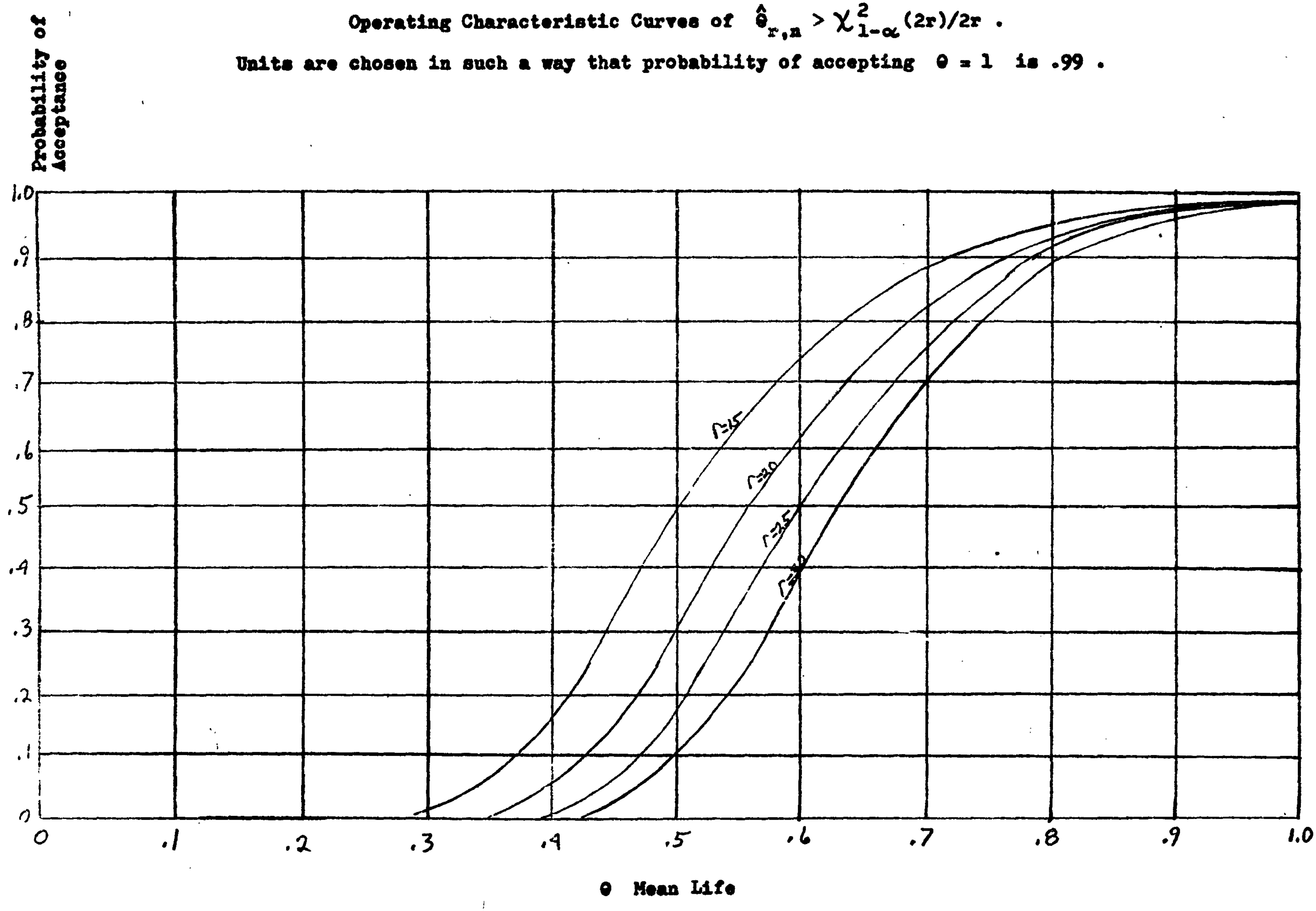

Figure . 4)

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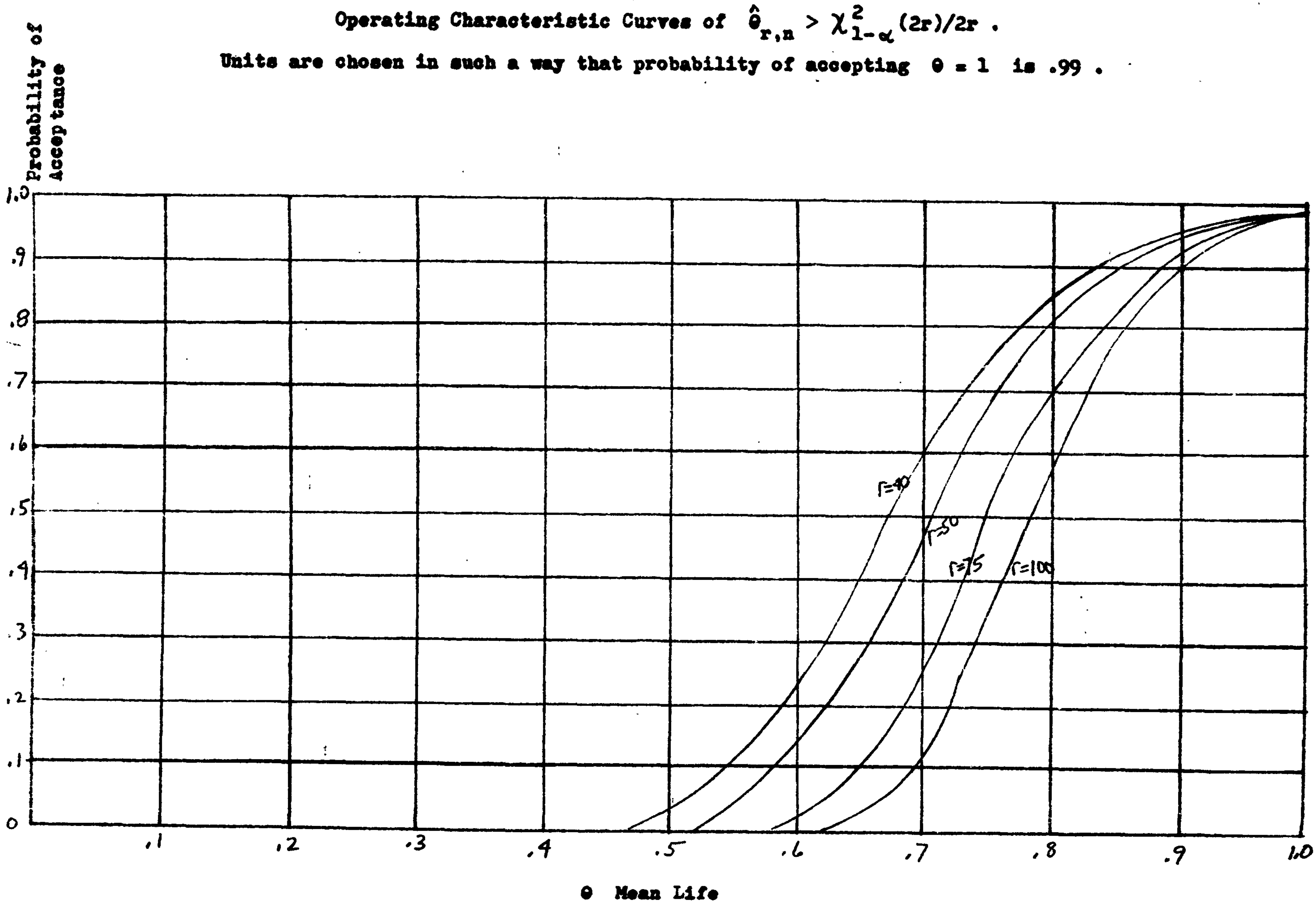

Figure 1(-1)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .95 .

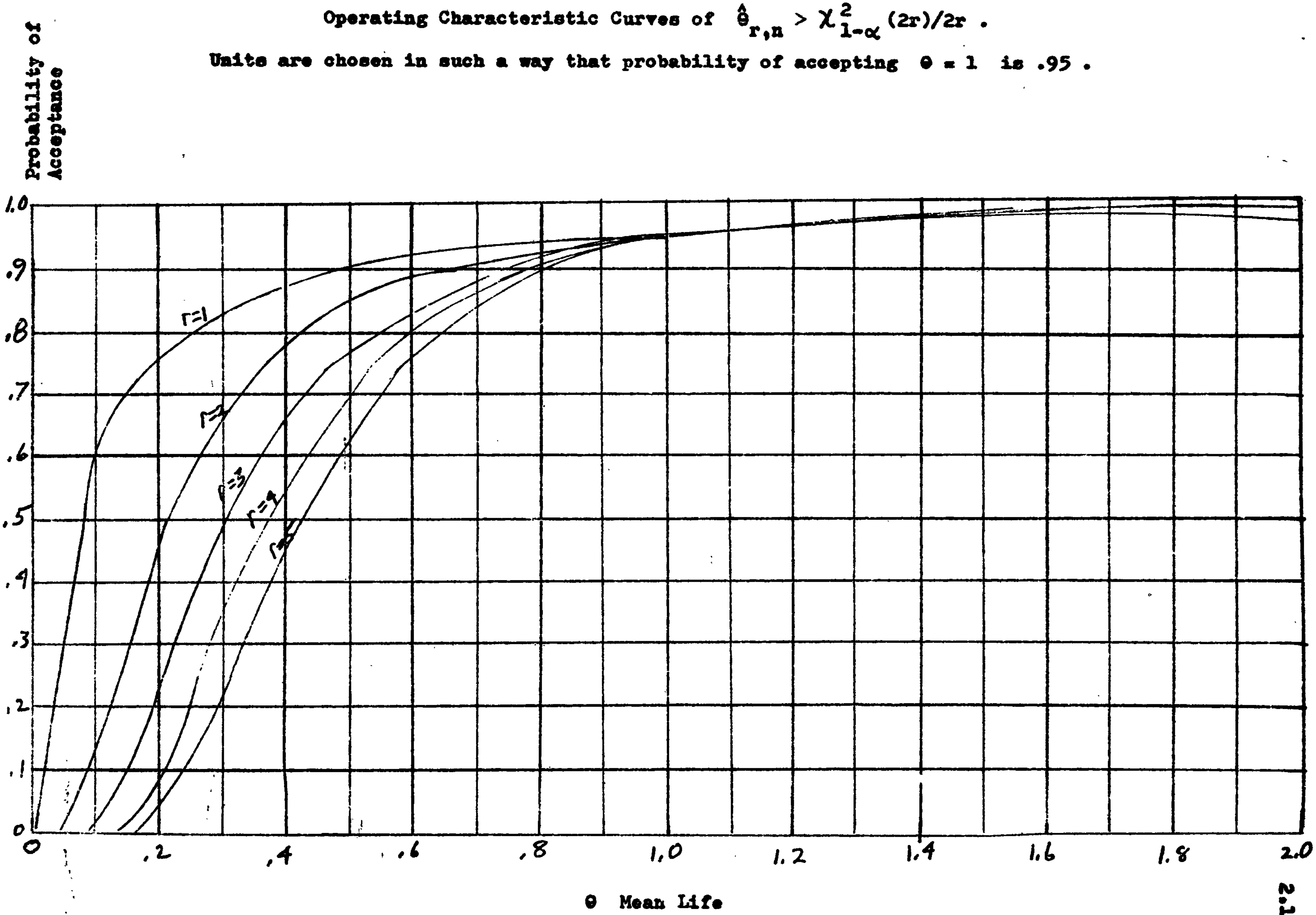

Figure 1(2)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

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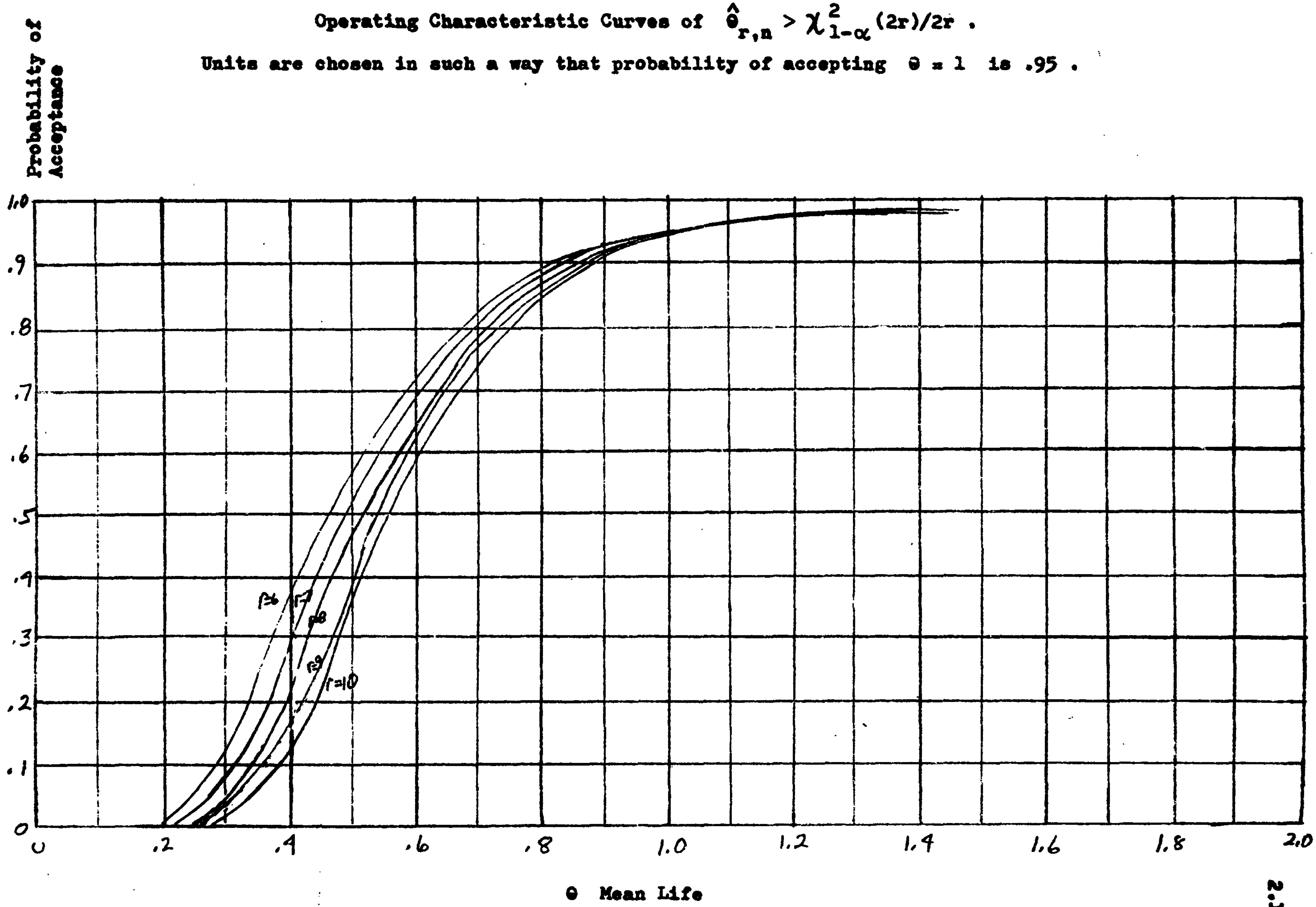

2.121

Figure (b$_3$)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

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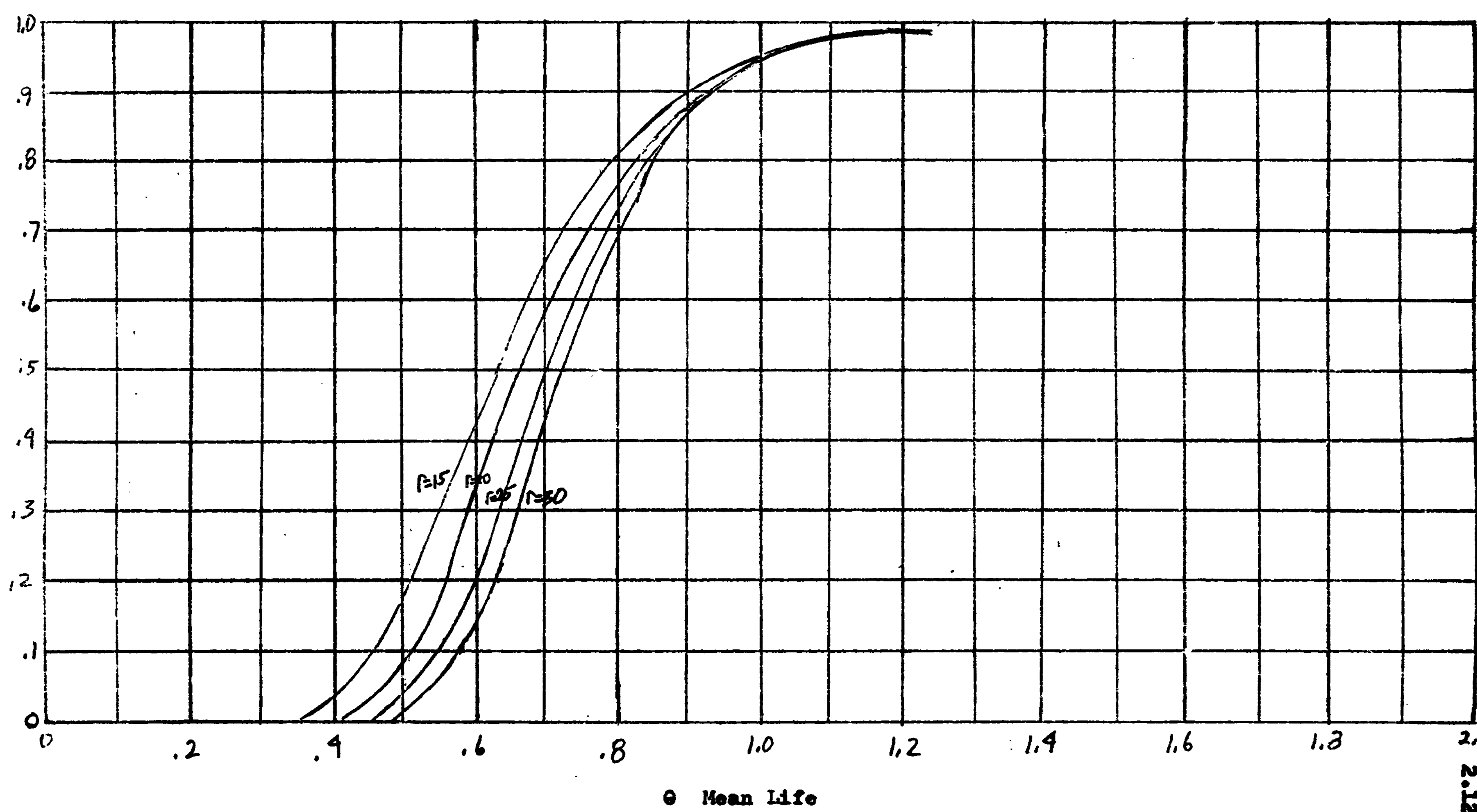

Figure 1(4)

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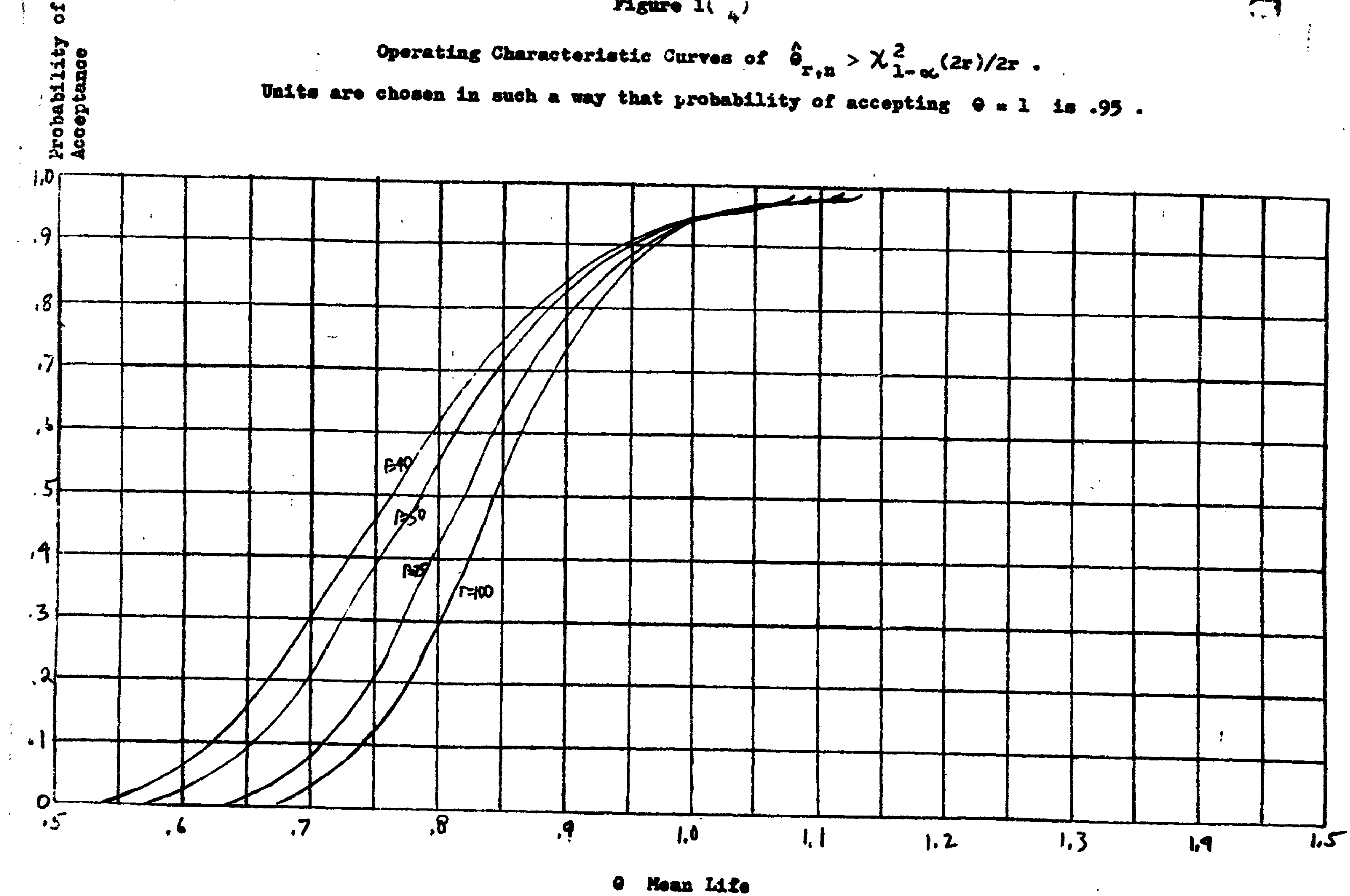

Figure [illegible]c_1)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .90 .

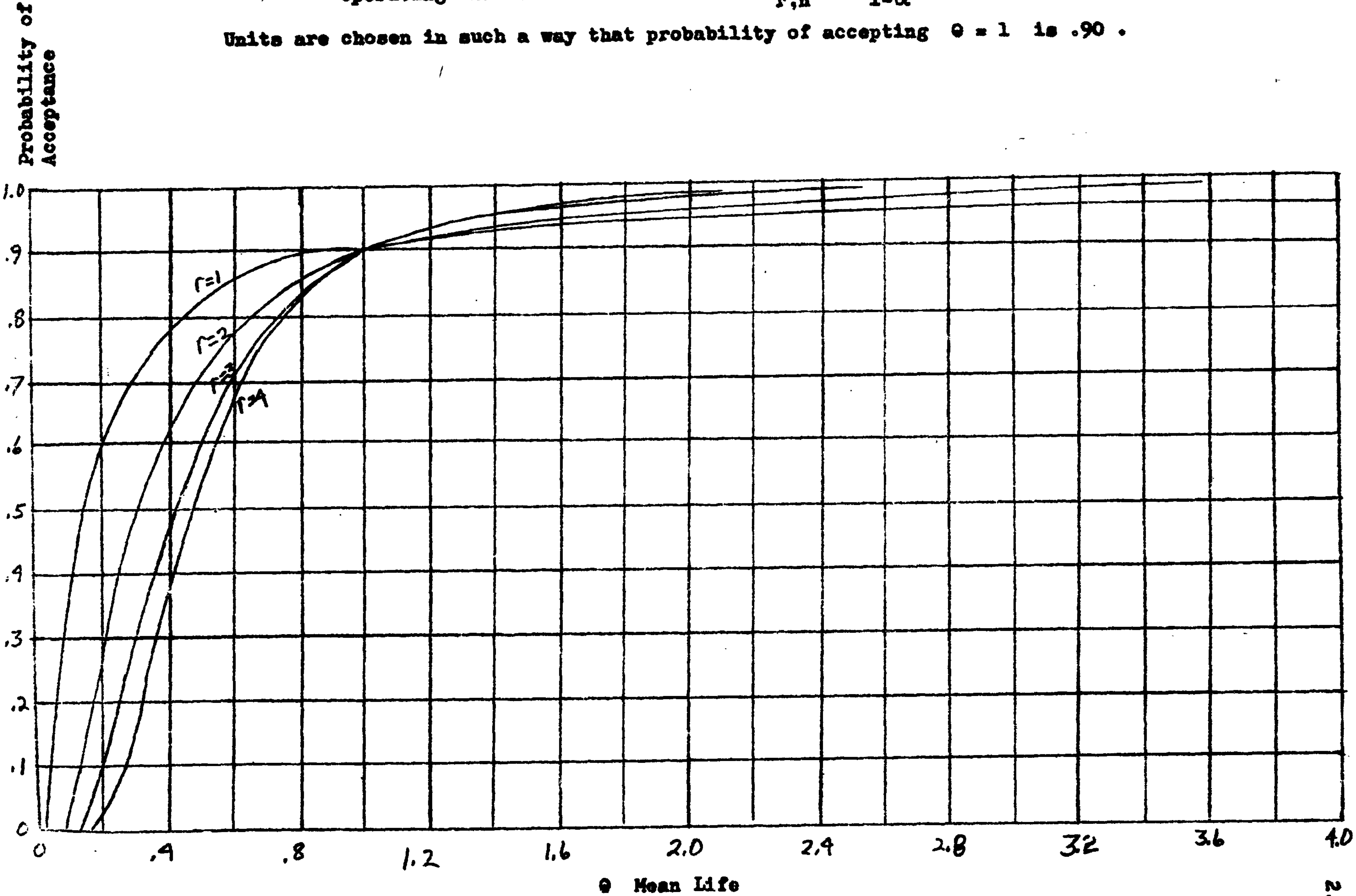

Figure 1(c)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

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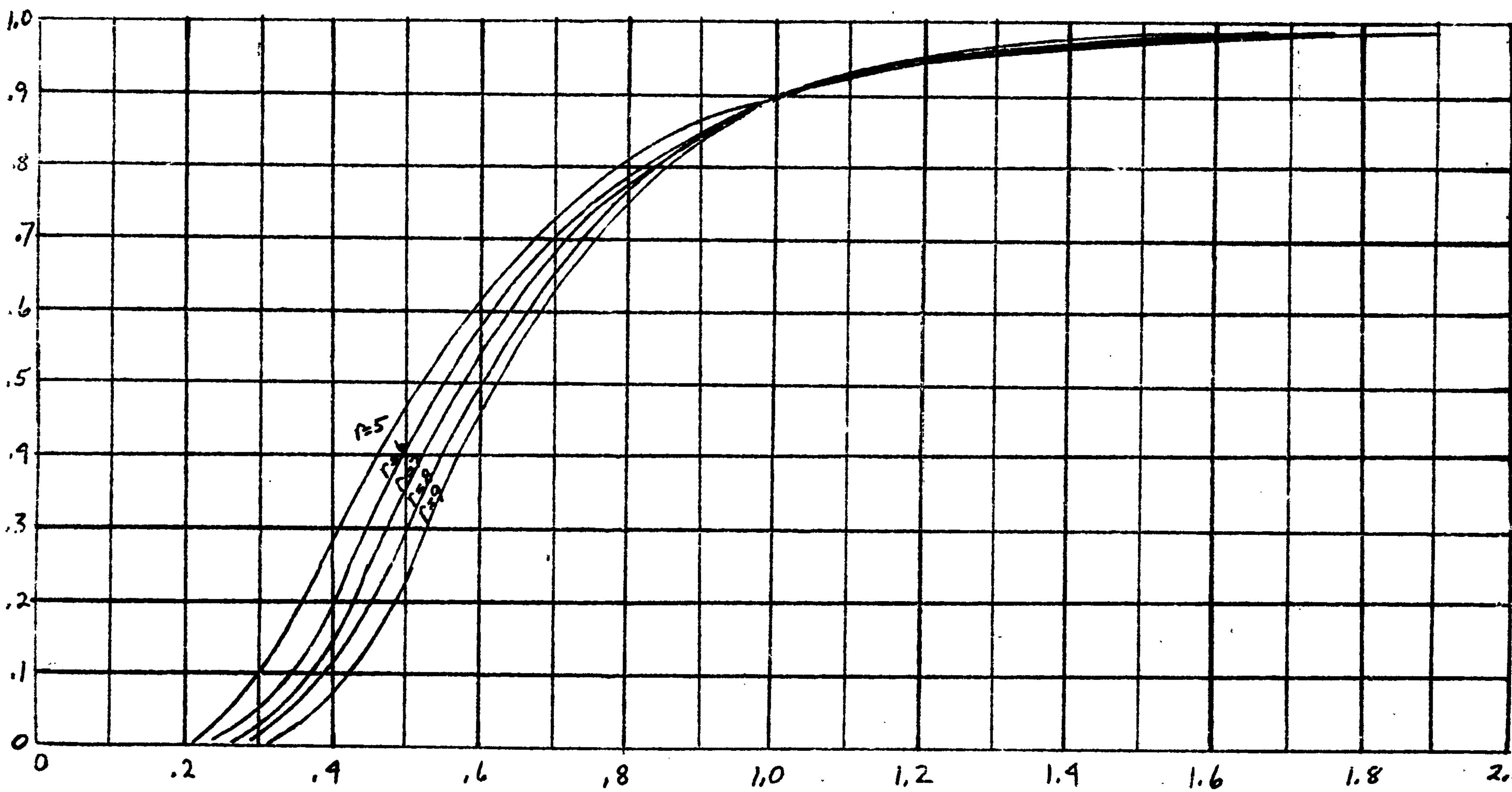
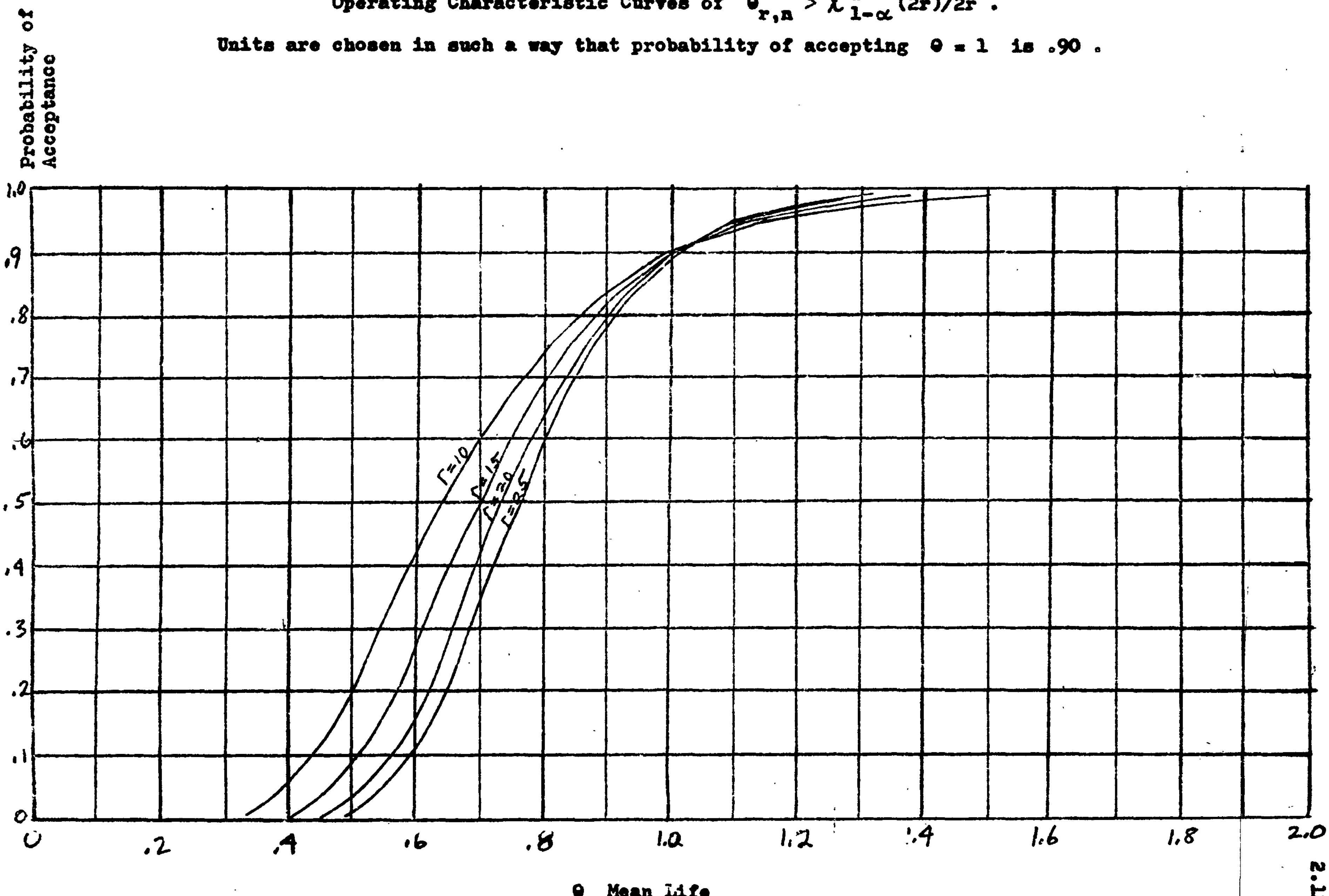


θ Mean Life

Figure (c3)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .90 .



2.126

Figure -(c$_4$)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

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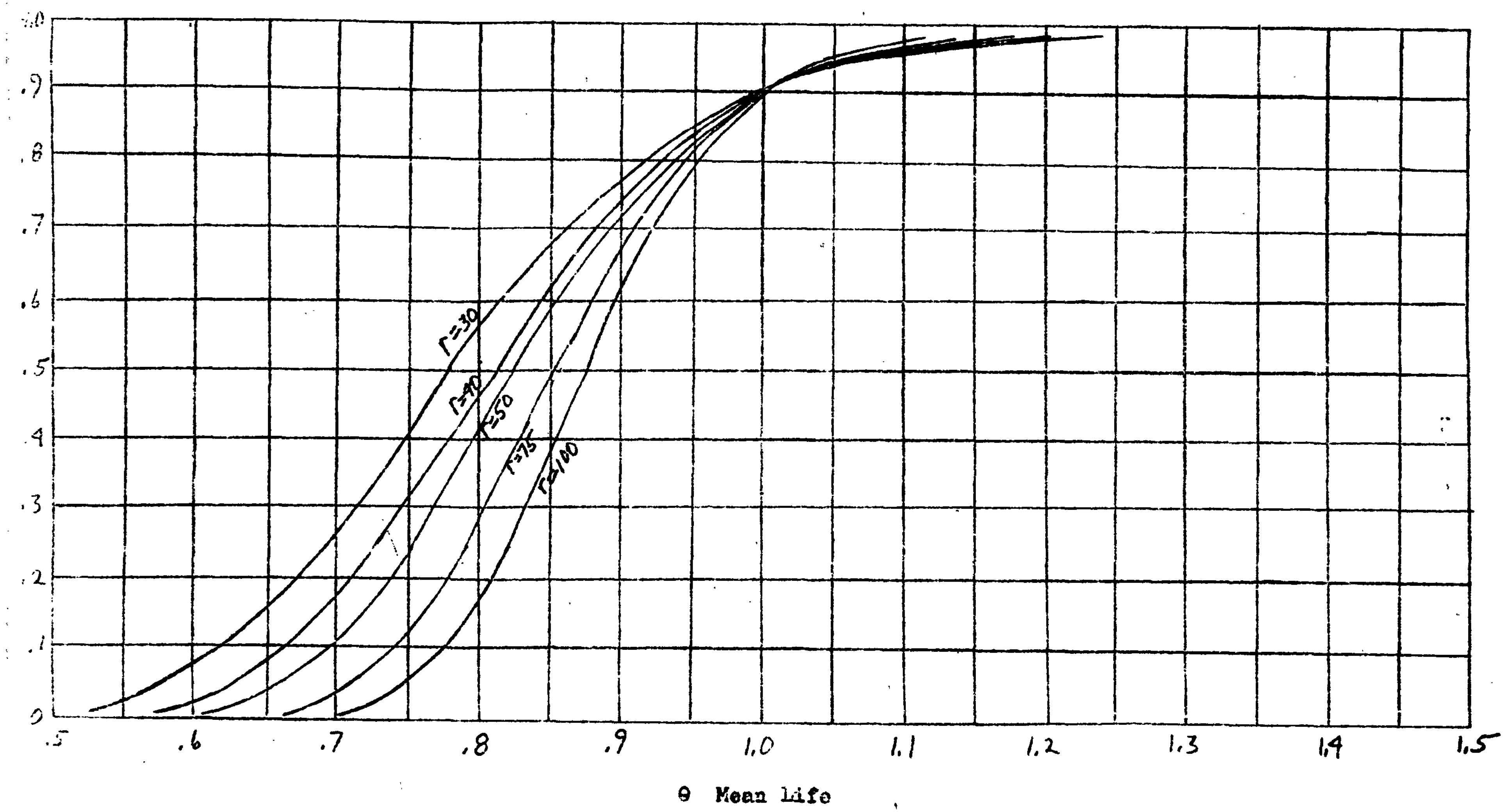
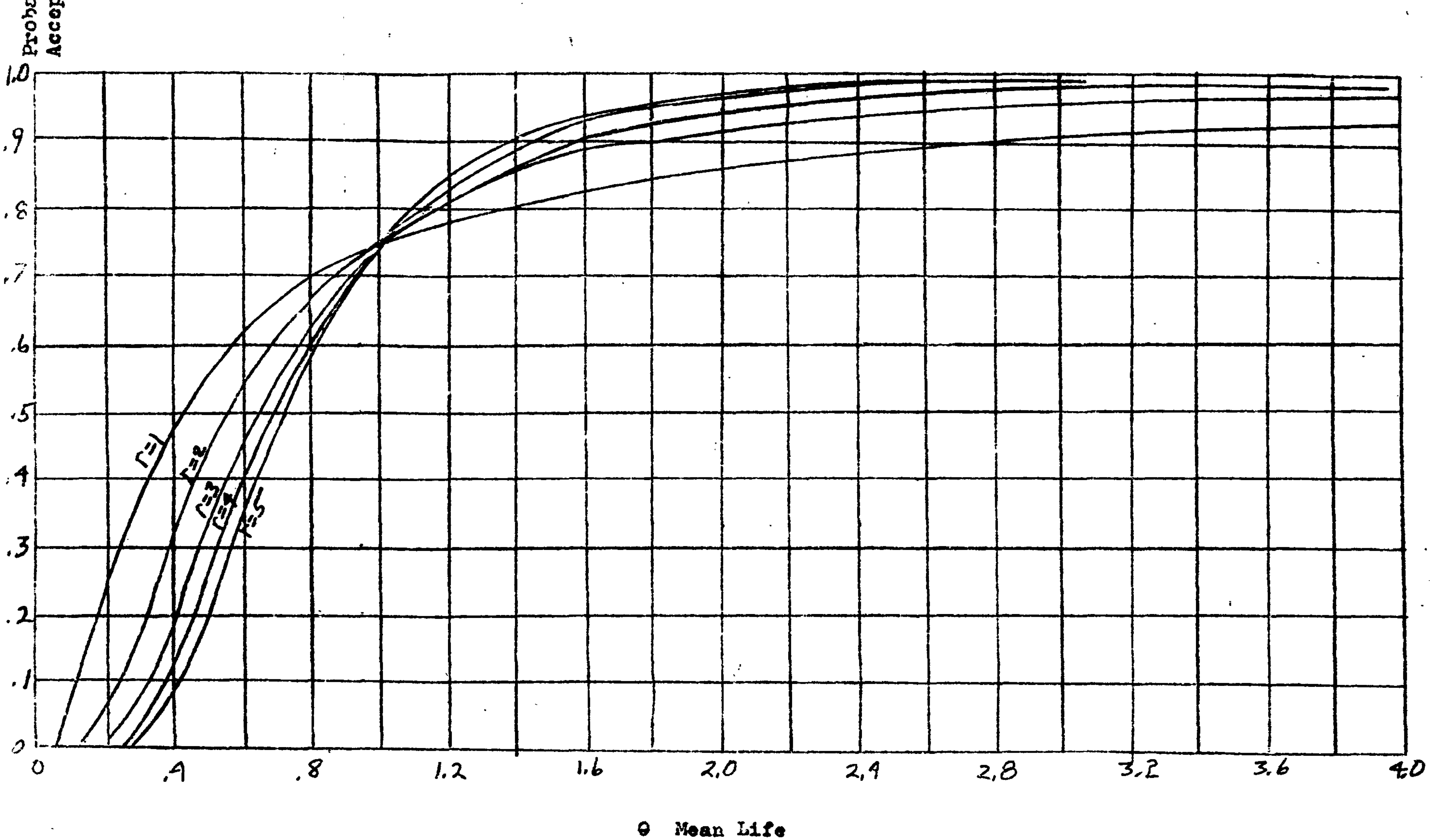


Figure - 1)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .75 .



Figu. 1(d_2)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .75 .

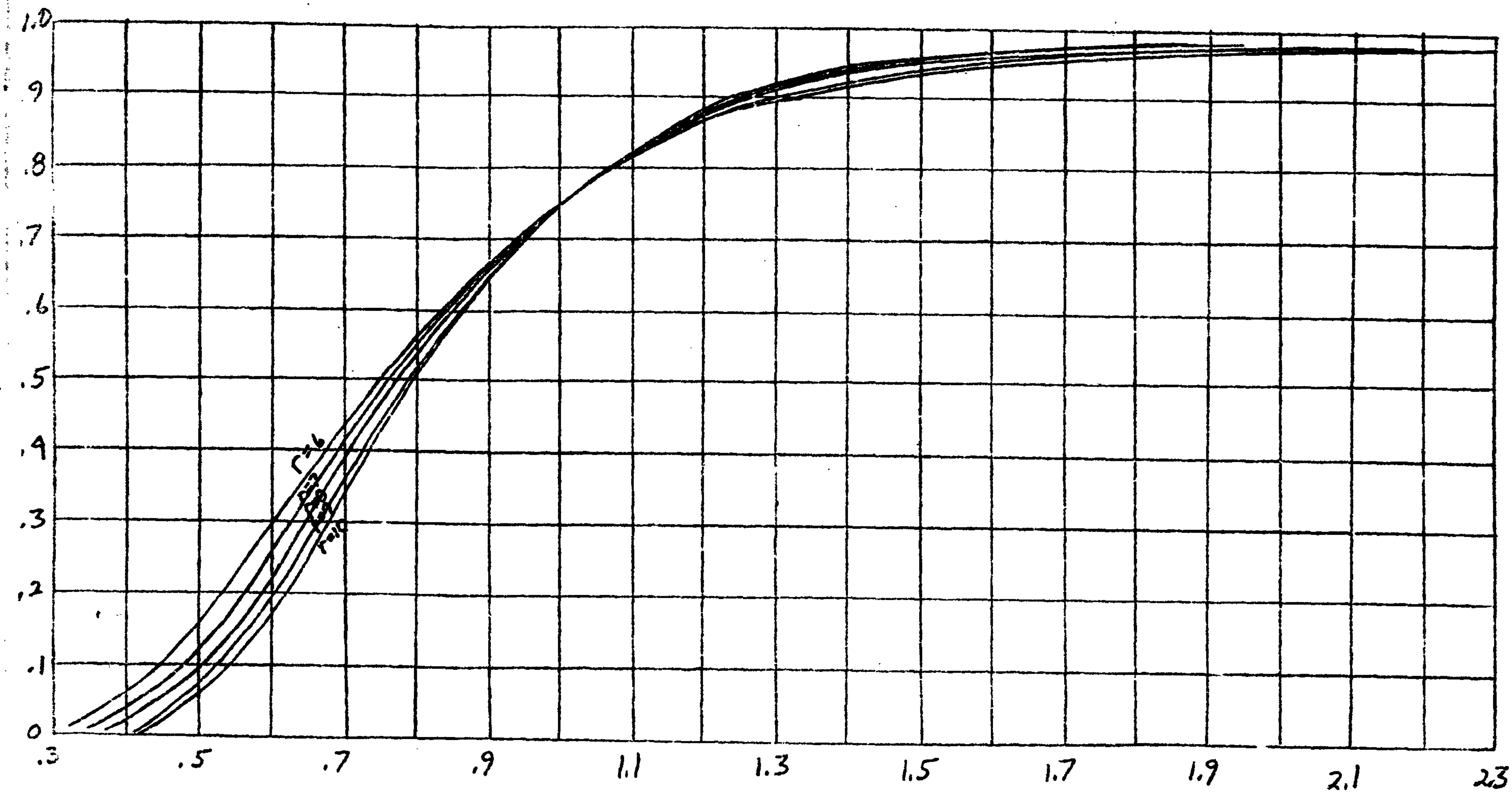


Figure 1(a_3)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .75 .

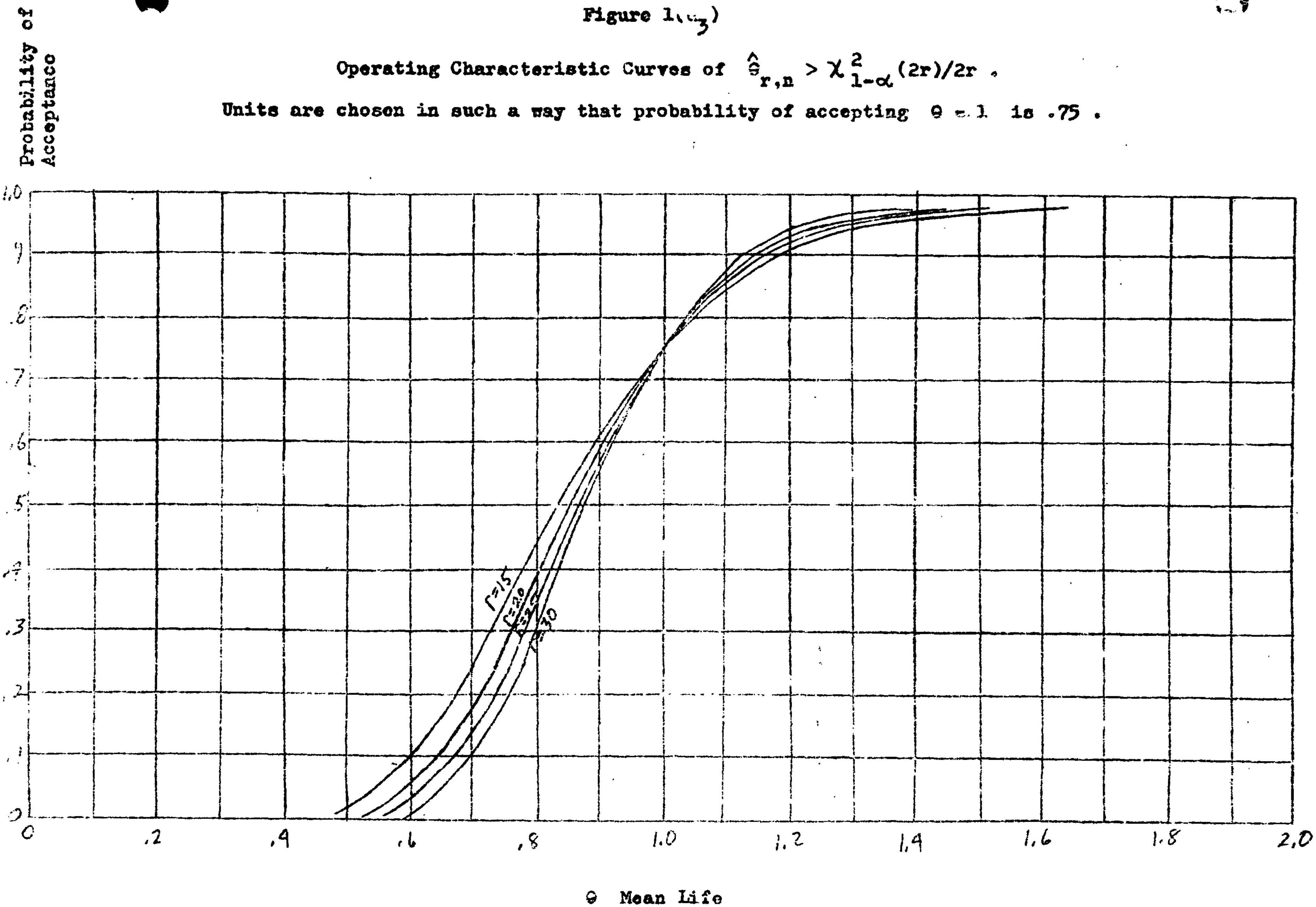


Figure 1(d_4)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .75 .

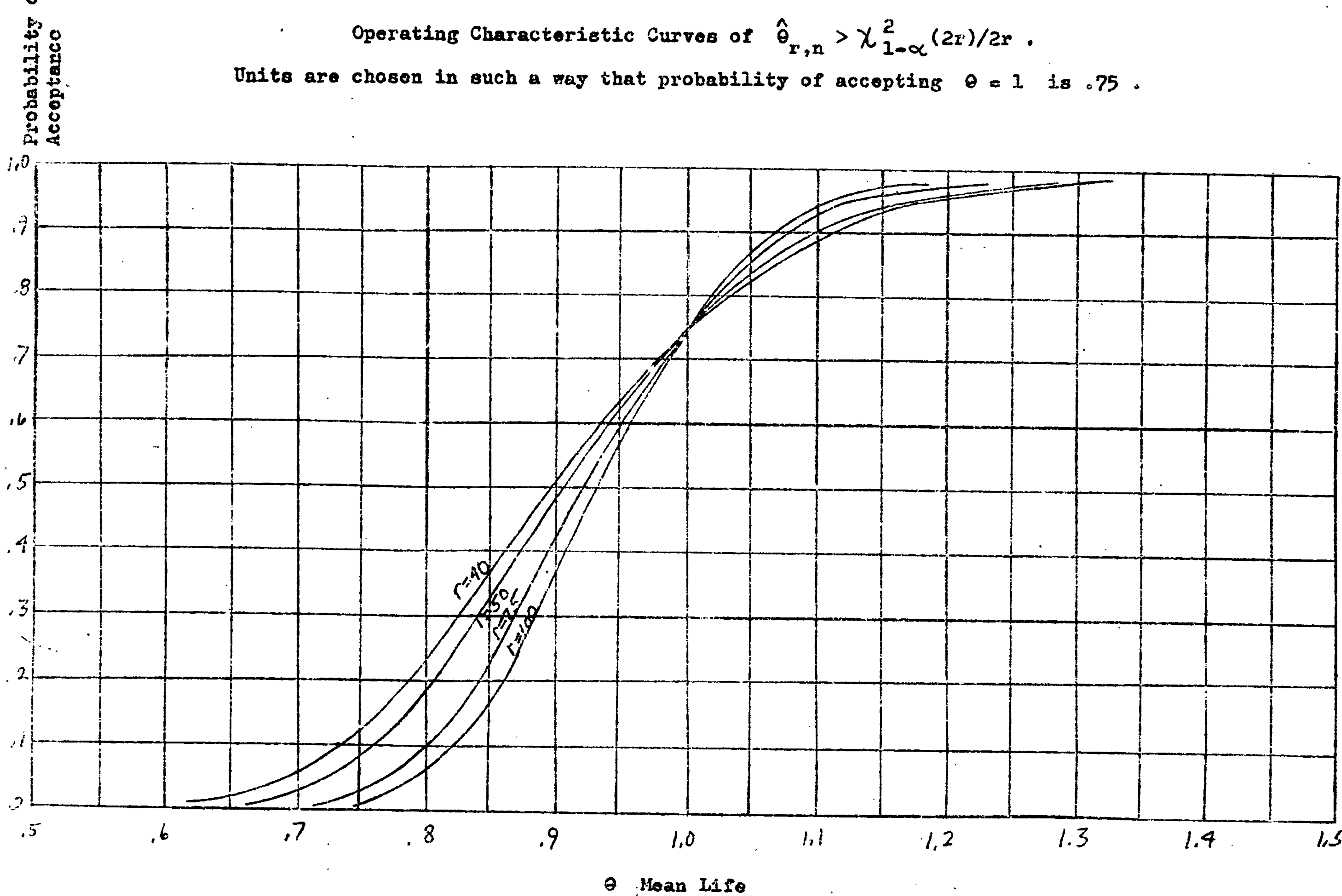


Figure 1(e_1)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .50 .

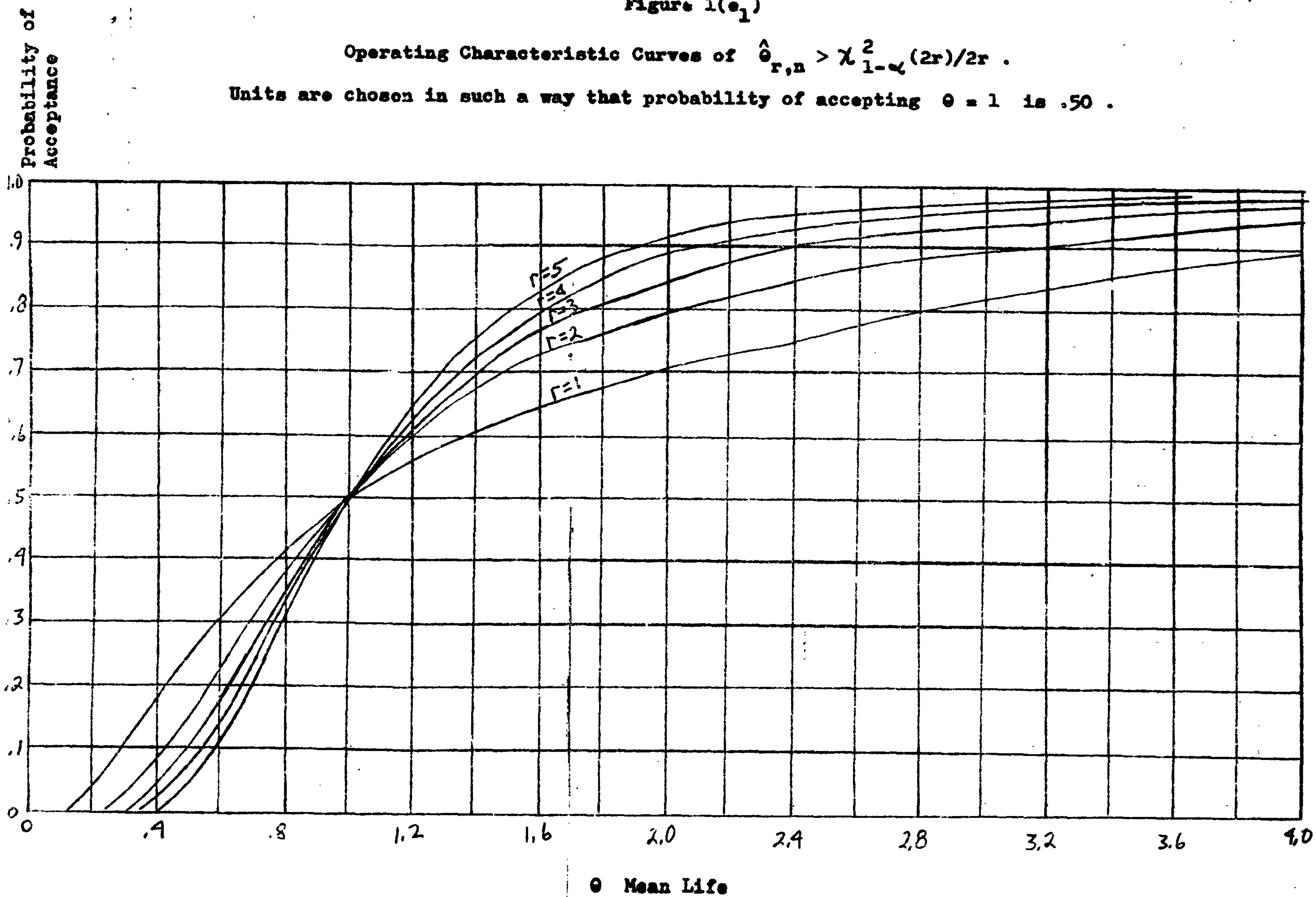


Figure -(e_2)

Operating characteristic curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

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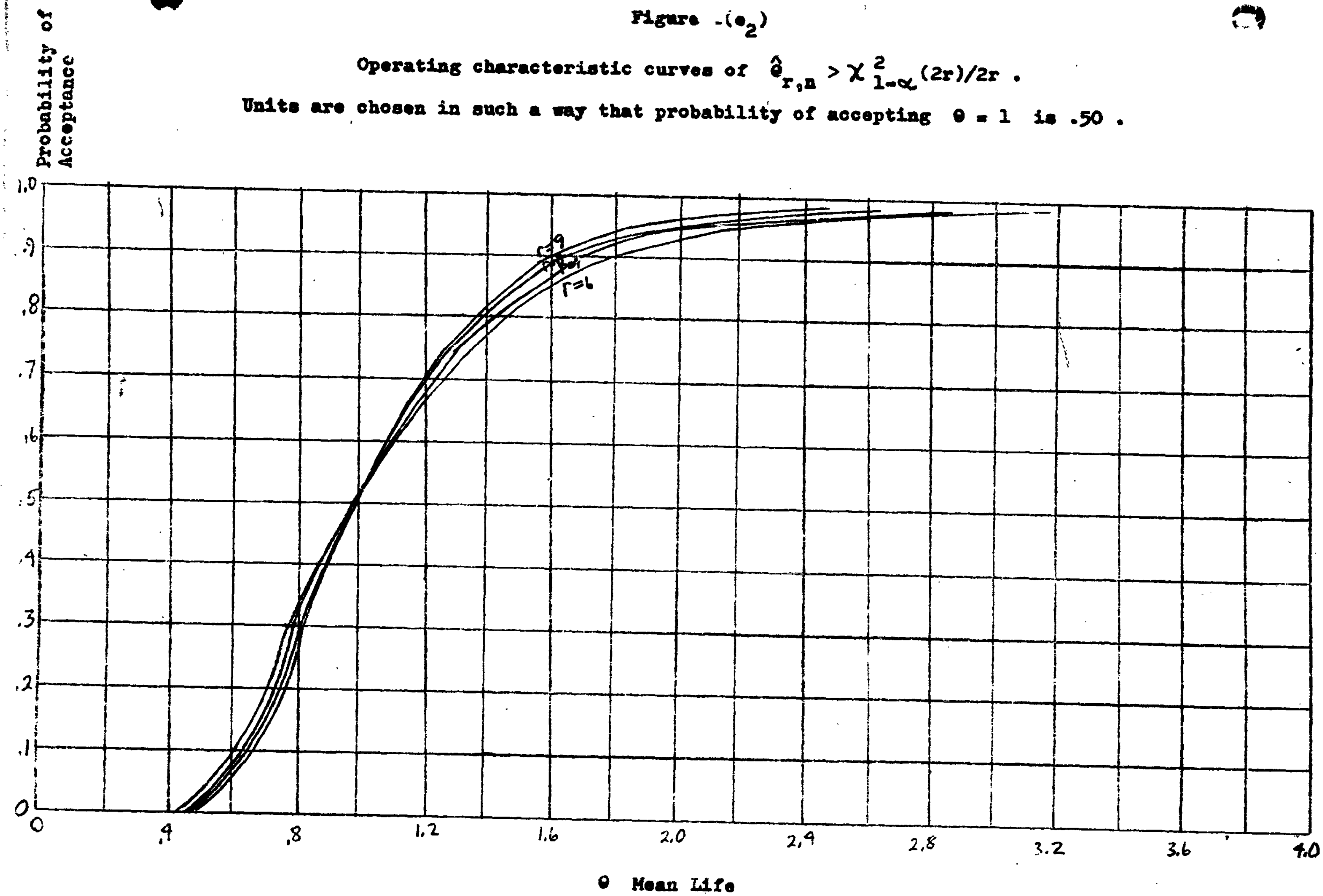


Figure (c3)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .50 .

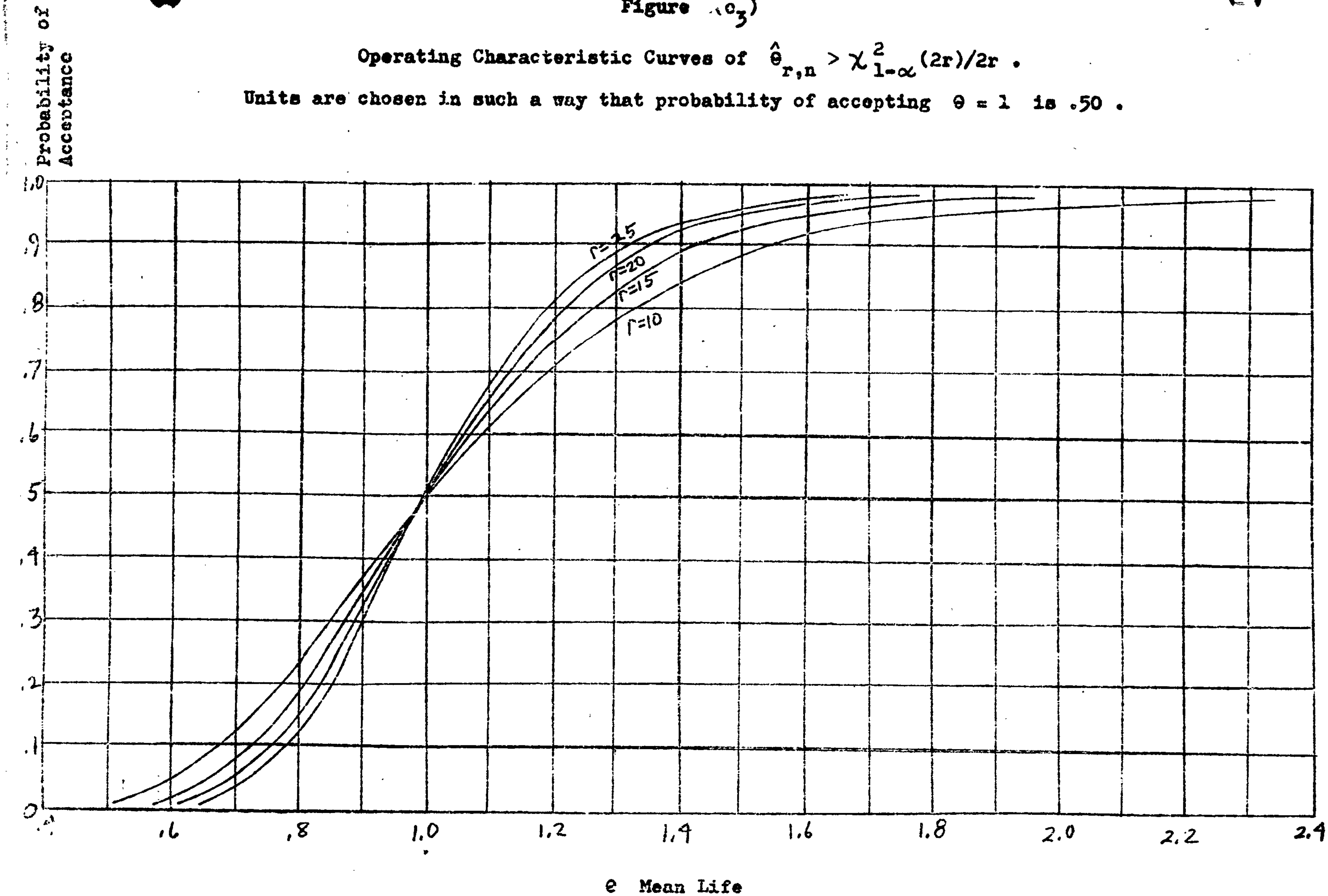


Figure 1(e_4)

Operating Characteristic Curves of $\hat{\theta}_{r,n} > \chi^2_{1-\alpha}(2r)/2r$.

Units are chosen in such a way that probability of accepting $\theta = 1$ is .50 .

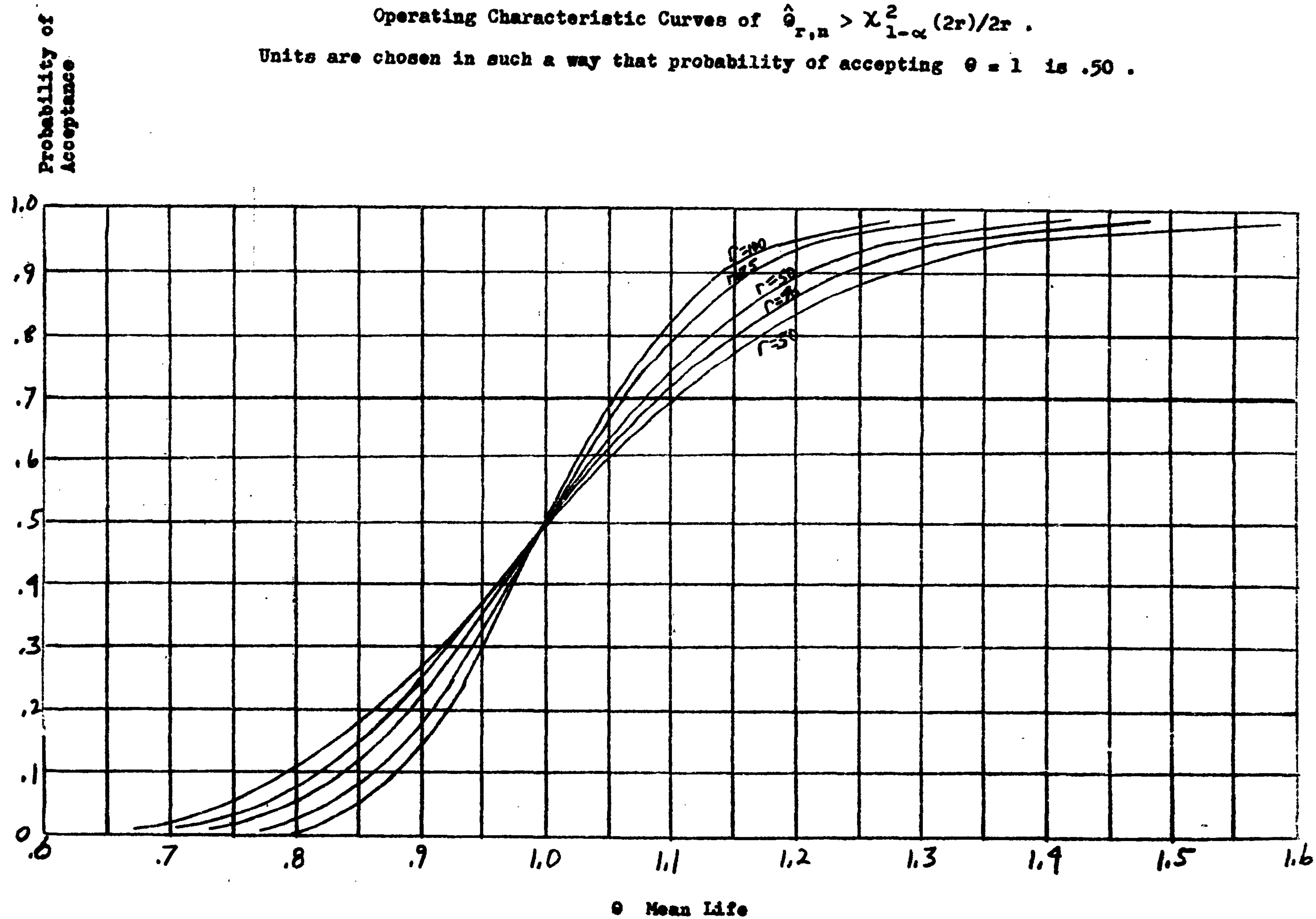


FIGURE 2

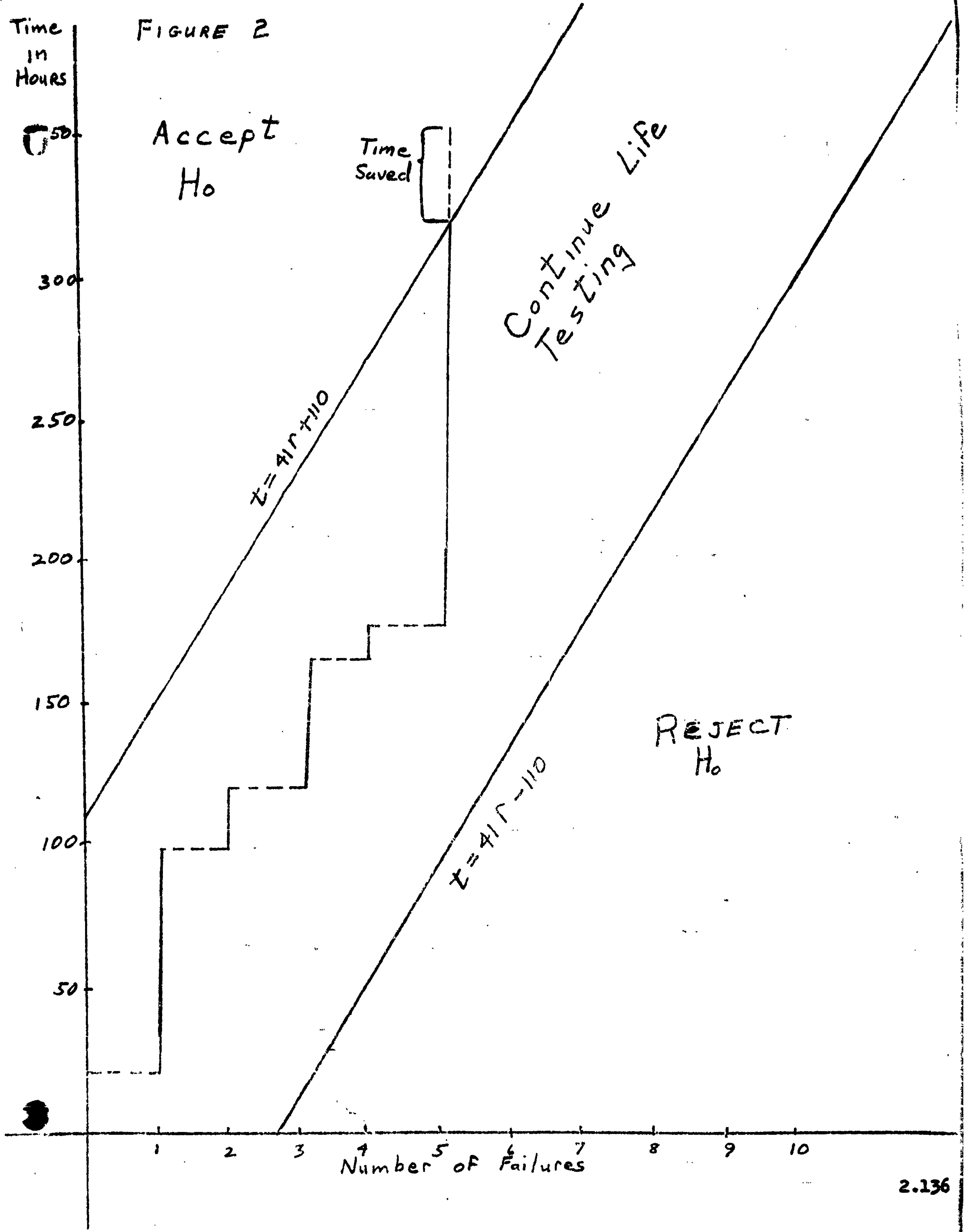

Time in Hours
350
300
250
200
150
100
50
Accept H₀
Time Saved
Continue Life Testing
t=41r+110
t=41r-110
Reject H₀
1
2
3
4
5
6
7
8
9
10
Number of Failures

FIGURE 3

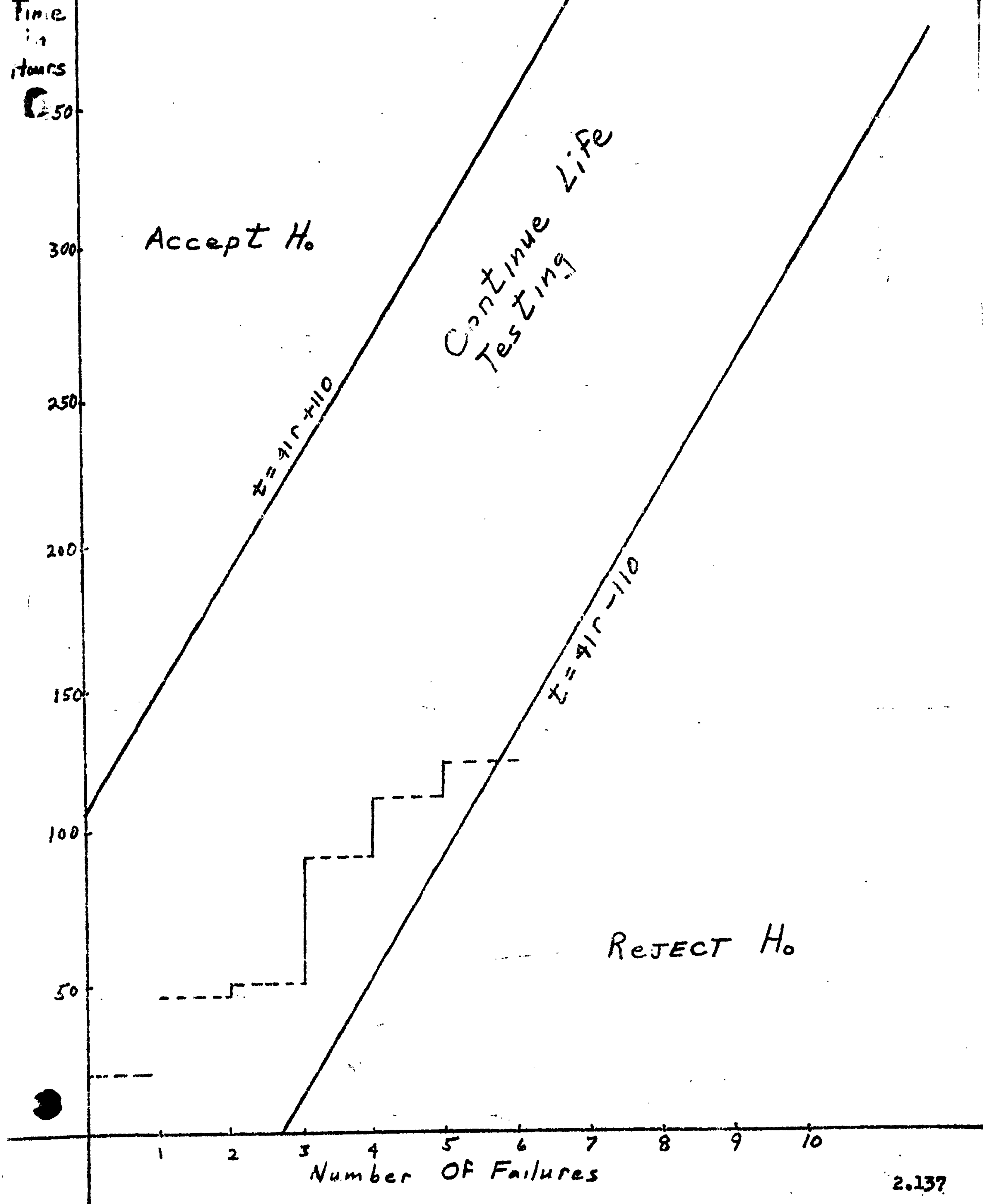
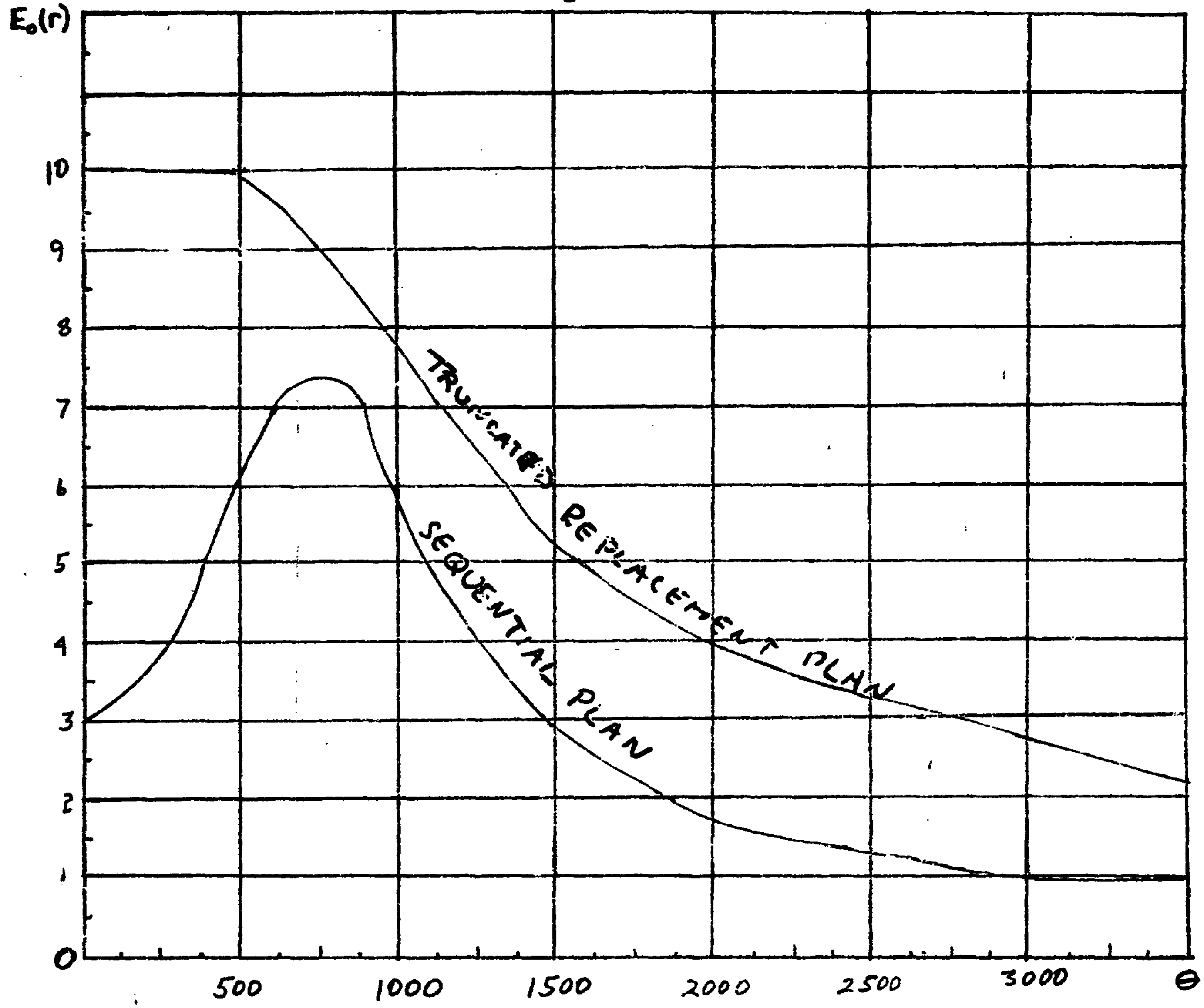

Time in Hours
350
300
250
200
150
100
50
Accept H_0
Continue Life Testing
$t = 41r + 110$
$t = 41r - 110$
Reject H_0
1
2
3
4
5
6
7
8
9
10
Number Of Failures

Figure 4(a)



Comparison of $E_\theta(r)$ curves for sequential and truncated replacement plans. The O.C. curves for each plan are such that $L(\theta_o) = .95$ and $L(\theta_1) = .05$ with $\theta_o = 1500$ and $\theta_1 = 500$.

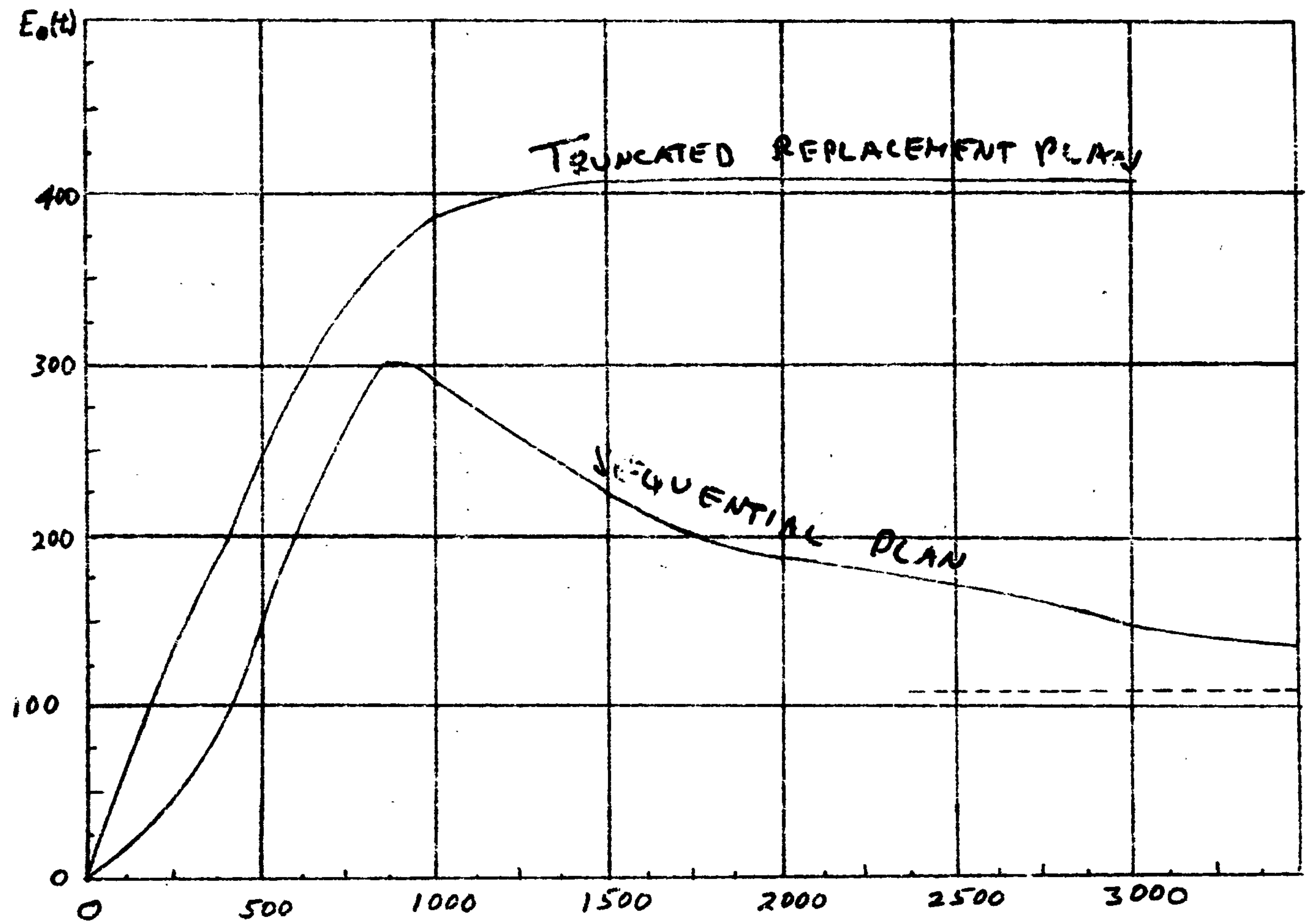


Figure 4(b)

Comparison of $E_\theta(t)$ curves for sequential and truncated replacement plans. The O.C. curves are such that $L(\theta_o) = .95$ and $L(\theta_1) = .05$, with $\theta_o = 1500$ and $\theta_1 = 500$. The 110 dashed line gives the value which $E_\theta(t)$ approaches asymptotically as $\theta \longrightarrow \infty$.

The TRUNCATION ON $V(t)$ induced by truncating the life test at $r=r_0$

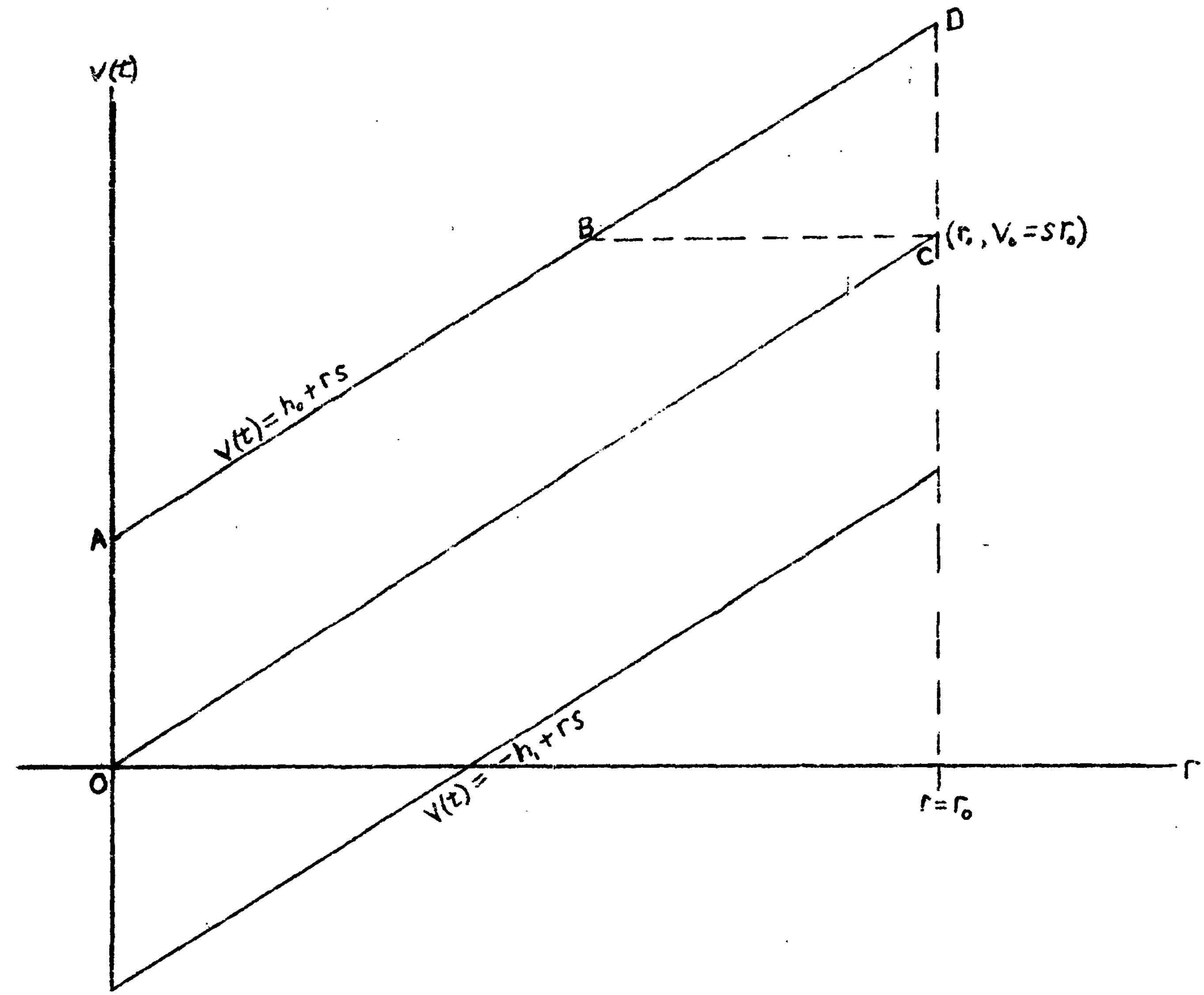


FIGURE 5

The truncation on r induced by truncating the life test at $V(t) = V_0$

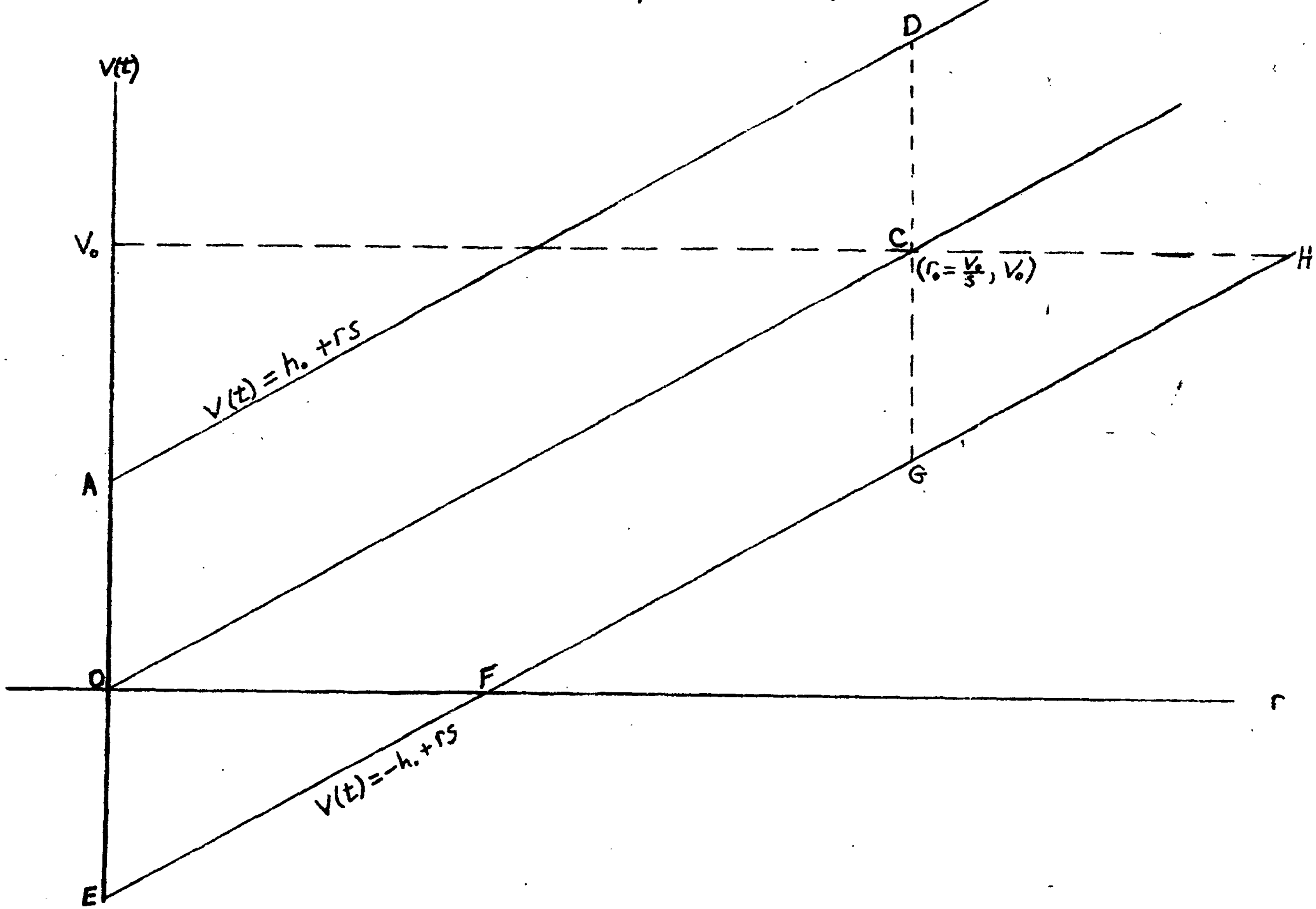
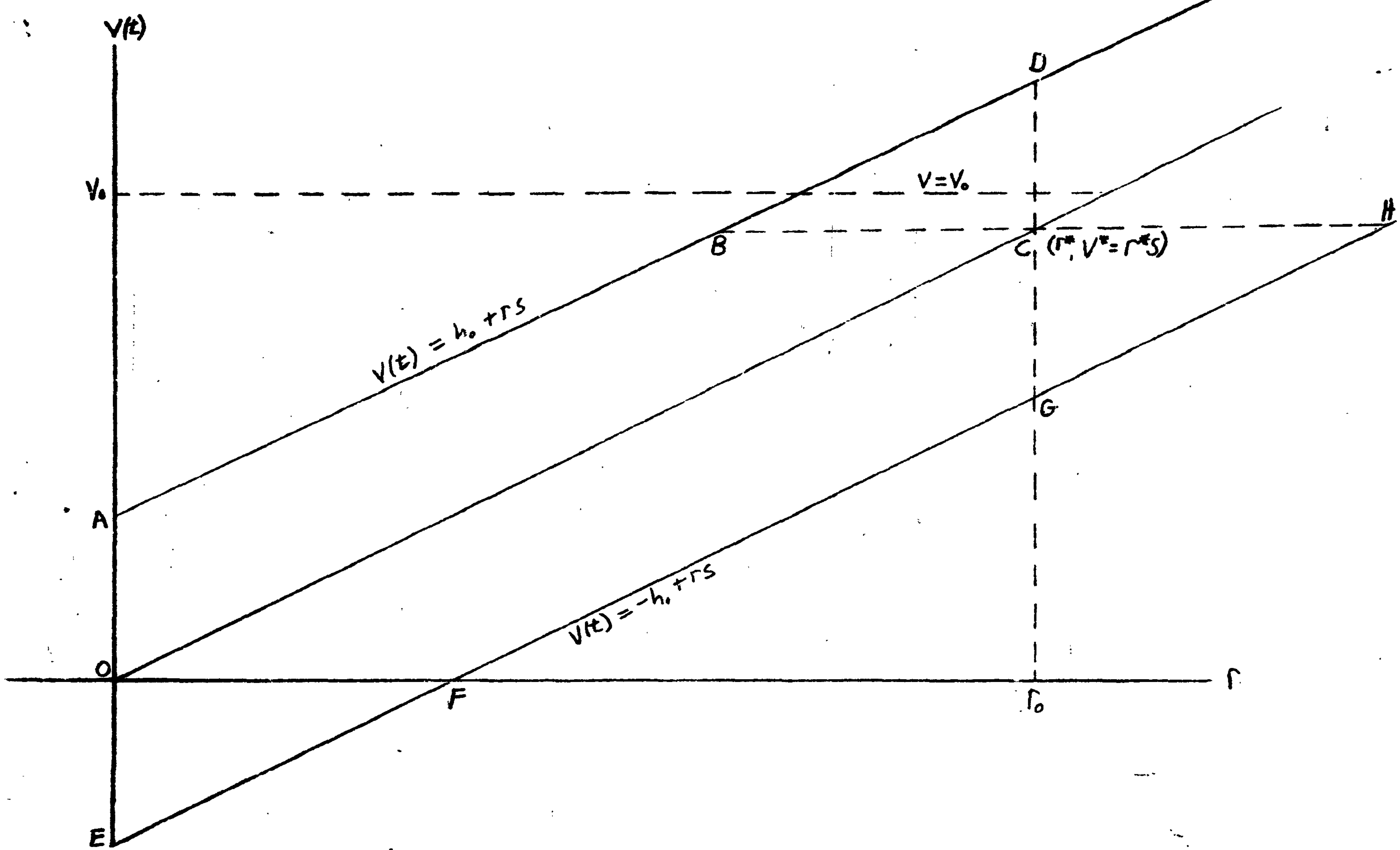


FIGURE 6

The Truncated Life Test induced by truncating at $r=r_0$ and $V=V_0$



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